Evolution of Mixed Dirac Particles Interacting with an External Magnetic Field

Maxim Dvornikov[a,b*] and Jukka Maalampi[a,c†]
[a] *Department of Physics, P.O. Box 35, FIN-40014 University of Jyväskylä, Finland;*
[b] *IZMIRAN, 142190, Troitsk, Moscow region, Russia;*
[c] *Helsinki Institute of Physics, P.O. Box 64, FIN-00014 University of Helsinki, Finland*

(Dated: July 16, 2007)

We study in the framework of relativistic quantum mechanics the evolution of a system of two Dirac neutrinos that mix with each other and have non-vanishing magnetic moments. The dynamics of this system in an external magnetic field is determined by solving the Dirac-Pauli equation with a given initial condition. Particularly we consider neutrino spin-flavor oscillations in a constant magnetic field and derive the analytical expression for the transition probability for the spin-flavor conversion in the limit where the energy associated with the magnetic interaction arising from transition magnetic moment is small compared with the total neutrino energy. Our new result exactly takes into account an external magnetic field and the parameters of neutrino system. Although we consider neutrinos, our formalism is straightforwardly applicable to any spin-1/2 particles.

PACS numbers: 14.60.Pq, 14.60.St, 03.65.Pm
Keywords: particle mixing, magnetic field, neutrino spin-flavor oscillations

In the neutrino oscillation phenomena observed so far oscillations occur between different neutrino flavors ν_ℓ ($\ell = e,\mu,\tau$) so that an active left-handed neutrino goes to another active left-handed neutrino (e.g. $\nu_e^{\mathrm{L}} \leftrightarrow \nu_\mu^{\mathrm{L}}$). In some situations another type of oscillation is possible, namely the so called spin-oscillation, where oscillation happens between an active left-handed neutrino and its inert right-handed counterpart (e.g. $\nu_e^{\mathrm{L}} \leftrightarrow \nu_e^{\mathrm{R}}$). In addition, also a combination of these two oscillation types can happen. This kind of oscillations are called spin-flavor oscillations. There oscillations take place between an active left-handed neutrino and an inert right-handed neutrino of different flavor (e.g. $\nu_e^{\mathrm{L}} \leftrightarrow \nu_\mu^{\mathrm{R}}$) [1].

In this paper we shall consider spin-flavor oscillations in an external magnetic field. We assume that neutrinos are Dirac particles with non-vanishing magnetic moments. We shall use the formalism of relativistic quantum mechanics, that is, we use the Dirac equation as a starting point, which is a proper approach for spin-1/2 particles. One of us (MD) has used this formalism for studying neutrino flavor oscillations in vacuum [2] and in an external axial-vector field [3] (neutrino interaction with matter). Neutrino spin-flavor oscillations in electromagnetic fields of various configurations were studied in Refs. [4]. The present paper is a continuation of these previous works.

Let us consider a system of two Dirac neutrinos with non-vanishing masses and magnetic moments. In general, both the mass matrix and the matrix of magnetic moments are non-diagonal in the flavor basis of neutrino wave functions. When the mass part of the Hamiltonian is diagonalized by a unitary transformation and the original flavor basis is replaced by a new set of wave functions in the mass eigenstate basis, the matrix of magnetic moments is generally not diagonal in the new basis. This means that there will be transition magnetic moments between the mass eigenstates. We will discuss the situation where the resulting magnetic moment matrix is close to diagonal, i.e. the transition magnetic moments are small compared with the diagonal magnetic moments in the mass eigenstate basis. When this is the situation one can apply the formalism developed in Refs. [2].

Assuming ultra-relativistic initial neutrinos and a magnetic field transversal with respect to the neutrino momentum, we will derive the leading (in transition magnetic moment) and the next-to-leading contributions to the transition probability for the process like $\nu_\beta^{\mathrm{L}} \to \nu_\alpha^{\mathrm{R}}$ (α,β denote two different flavors) in the flavor eigenstate basis.

Let us denote the magnetic moments of the two Dirac neutrinos ν_α,ν_β as $M_{\ell\ell'}$. The off-diagonal elements of the matrix, $(M_{\ell\ell'})$, are called transition magnetic moments. The Lagrangian of the neutrinos in the presence of an external electromagnetic field $F_{\mu\nu} = (\mathbf{E},\mathbf{B})$ is given by

$$\mathcal{L}(\nu_\alpha,\nu_\beta) = \sum_{\ell=\alpha,\beta} \mathcal{L}_0(\nu_\ell) - (m_{\beta\alpha}\bar{\nu}_\beta\nu_\alpha + \mathrm{h.c.}) - \frac{1}{2}\sum_{\ell\ell'=\alpha,\beta} M_{\ell\ell'}\bar{\nu}_\ell\sigma_{\mu\nu}\nu_{\ell'}F^{\mu\nu}. \tag{1}$$

Here, e.g., $\mathcal{L}_0(\nu_\ell) = \bar{\nu}_\ell(\mathrm{i}\gamma^\mu\partial_\mu - m_{\ell\ell})\nu_\ell$, $m_{\ell\ell}$ and $m_{\beta\alpha}$ (a vacuum mixing parameter) have dimension of mass, and $\sigma_{\mu\nu} = (\mathrm{i}/2)[\gamma_\mu,\gamma_\nu]$.

*Electronic address: maxim.dvornikov@phys.jyu.fi
†Electronic address: jukka.maalampi@phys.jyu.fi

To analyze the dynamics of the system one has to set the initial condition by giving the initial wave functions of neutrinos ν_ℓ and then analytically determine the field distributions at subsequent moments of time. The analogous problem was studied in Refs. [2] for mixed neutrinos in vacuum and in Ref. [3] for neutrinos with mixing in an external axial-vector field.

We assume an initial condition of the form

$$\nu_\alpha(\mathbf{r},0)=0,\quad \nu_\beta(\mathbf{r},0)=\xi(\mathbf{r}), \tag{2}$$

where $\xi(\mathbf{r})$ is a function to be specified. If we identify ν_α as ν_e or ν_τ and ν_β as ν_μ, for example, this initial condition might correspond a situation where the source of neutrinos consists of pions and kaons decaying to muon neutrinos. With the opposite identification it could describe a situation where the source produces electron neutrinos only.

In order to eliminate the vacuum mixing term in Eq. (1) we introduce a new basis of the wave functions, the mass eigenstate basis ψ_a, $a=1,2$, obtained from the original flavor basis ν_ℓ through the unitary transformation

$$\nu_\ell=\sum_{a=1,2}U_{\ell a}\psi_a, \tag{3}$$

where the matrix $(U_{\ell a})$ is parametrized in terms of a mixing angle θ in the usual manner

$$(U_{\ell a})=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}. \tag{4}$$

The Lagrangian (1) rewritten in terms of the fields ψ_a takes the form

$$\mathcal{L}(\psi_1,\psi_2)=\sum_{a=1,2}\mathcal{L}_0(\psi_a)-\frac{1}{2}\sum_{ab=1,2}\mu_{ab}\bar{\psi}_a\sigma_{\mu\nu}\psi_b F^{\mu\nu}, \tag{5}$$

where $\mathcal{L}_0(\psi_a)=\bar{\psi}_a(\mathrm{i}\gamma^\mu\partial_\mu-m_a)\psi_a$ is the Lagrangian for the free fermion ψ_a with the mass m_a and

$$\mu_{ab}=\sum_{\ell\ell'=\alpha,\beta}U^{-1}_{a\ell}M_{\ell\ell'}U_{\ell' b}, \tag{6}$$

is the magnetic moment matrix presented in the mass eigenstates basis. Using Eqs. (2)-(4) the initial conditions for the fermions ψ_a become

$$\psi_1(\mathbf{r},0)=\sin\theta\xi(\mathbf{r}),\quad \psi_2(\mathbf{r},0)=\cos\theta\xi(\mathbf{r}). \tag{7}$$

Let us assume that the magnetic field is constant, uniform and directed along the z-axis, $\mathbf{B}=(0,0,B)$, and that the electric field vanishes, $\mathbf{E}=0$. In this case we write down the Dirac-Pauli equation for ψ_a, resulting from Eq. (5), as follows:

$$\mathrm{i}\dot{\psi}_a=\mathcal{H}_a\psi_a+V\psi_b,\quad a,b=1,2,\quad a\neq b, \tag{8}$$

where $\mathcal{H}_a=(\boldsymbol{\alpha}\mathbf{p})+\rho_3 m_a-\mu_a\rho_3\Sigma_3 B$ is the Hamiltonian for the particle ψ_a accounting for the magnetic field, $V=-\mu\rho_3\Sigma_3 B$ describes the interaction of the transition magnetic moment with the external magnetic field, $\mu_a=\mu_{aa}$, and $\mu=\mu_{ab}=\mu_{ba}$ are elements of the matrix (μ_{ab}). Here we use the usual definitions for the Dirac matrixes $\boldsymbol{\alpha}=\gamma^0\boldsymbol{\gamma}$, $\rho_3=\gamma^0$ and $\boldsymbol{\Sigma}=\gamma^0\gamma^5\boldsymbol{\gamma}$. It is in order to notice that a wave equation for a neutrino with the anomalous magnetic moment that differs from (8) was derived in Ref. [5].

Imposing the initial condition (7), we now solve Eq. (8) perturbatively. We assume that the characteristic energy of the particle ψ_a is in the absence of the flavor changing magnetic interaction $V\psi_b$ (i.e. the transition magnetic moment $\mu=0$) much greater than $|\mu B|$. We present the solution of Eq. (8) in the expansion form

$$\psi_a(\mathbf{r},t)=\psi_a^{(0)}(\mathbf{r},t)+\psi_a^{(1)}(\mathbf{r},t)+\dots, \tag{9}$$

where $\psi_a^{(0)}(\mathbf{r},t)$ is the solution of Eq. (8) without the the last term $V\psi_b$, that is the eigenvalue of the Hamiltonian $\mathcal{H}_a$, and $\psi_a^{(1)}(\mathbf{r},t)$ is a function linear in μB.

We first determine the function $\psi_a^{(0)}(\mathbf{r},t)$ (we follow here Ref. [6]). One can present the wave function $\psi_a^{(0)}$ as follows:

$$\psi_a^{(0)}(\mathbf{r},t)=\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}}e^{\mathrm{i}\mathbf{pr}}\sum_{\zeta=\pm1}\left[a_a^{(0)}(\zeta,\mathbf{p})u_a^{(\zeta)}\exp(-\mathrm{i}E_a^{(\zeta)}t)+b_a^{(0)}(\zeta,\mathbf{p})v_a^{(\zeta)}\exp(+\mathrm{i}E_a^{(\zeta)}t)\right],$$

where $a_a^{(0)}(\zeta,\mathbf{p})$ and $b_a^{(0)}(\zeta,\mathbf{p})$ are arbitrary coefficients, and $\zeta=\pm 1$ characterizes the direction of the particle spin with respect to the magnetic field. We notice that the polarization operator

$$\Pi_a = m_a\Sigma_3 + \rho_2[\mathbf{\Sigma}\times\mathbf{p}]_3 - \mu_a B,$$

where $\rho_2 = \mathrm{i}\gamma^0\gamma^5$, commutes with the Hamiltonian $\mathcal{H}_a$, i.e. $[\Pi_a,\mathcal{H}_a]=0$, and that the identity $\Pi_a^2 + p_3^2 = \mathcal{H}_a^2$ is satisfied. It follows that the basic spinors $u_a^{(\zeta)}$ obey the equation

$$\Pi_a u_a^{(\zeta)} = \zeta\sqrt{E_a^{(\zeta)2} - p_3^2}\,u_a^{(\zeta)}.$$

Analogous expression can be written for the spinor $v_a^{(\zeta)}$. The particle energy as a function of the momentum and the magnetic field strength can be found if we suppose that the equation $\mathcal{H}_a u_a^{(\zeta)} = E_a^{(\zeta)} u_a^{(\zeta)}$ (and $\mathcal{H}_a v_a^{(\zeta)} = -E_a^{(\zeta)} v_a^{(\zeta)}$ for negative energies) has a nontrivial solution. We end up with the following expression for $E_a^{(\zeta)}$ (see also Ref. [6]):

$$E_a^{(\zeta)} = \sqrt{p_3^2 + \mathcal{E}_a^{(\zeta)2}},\quad \mathcal{E}_a^{(\zeta)} = \mathcal{E}_a - \zeta\mu_a B,\quad \mathcal{E}_a = \sqrt{m_a^2 + p_1^2 + p_2^2}.$$

Following the common method used in derivating the solutions of the Dirac equation in vacuum, one obtains for the normalized basis spinors the expression

$$u_a^{(\zeta)} = \frac{1}{2\sqrt{E_a^{(\zeta)}}}\begin{pmatrix} \phi_a^+\alpha_a^+ \\ -\zeta\phi_a^-\alpha_a^- e^{\mathrm{i}\varphi} \\ \phi_a^+\alpha_a^- \\ \zeta\phi_a^-\alpha_a^+ e^{\mathrm{i}\varphi}\end{pmatrix},\quad v_a^{(\zeta)} = \frac{1}{2\sqrt{E_a^{(\zeta)}}}\begin{pmatrix} \phi_a^+\alpha_a^- \\ \zeta\phi_a^-\alpha_a^+ e^{\mathrm{i}\varphi} \\ -\phi_a^+\alpha_a^+ \\ \zeta\phi_a^-\alpha_a^- e^{\mathrm{i}\varphi}\end{pmatrix},$$

where

$$\phi_a^{\pm} = \sqrt{1\pm\zeta m_a/\mathcal{E}_a},\quad \alpha_a^{\pm} = \sqrt{E_a^{(\zeta)} \pm \zeta\mathcal{E}_a^{(\zeta)}},$$

and $\tan\varphi = p_2/p_1$.

Using the orthonormality conditions for the spinors $u_a^{(\zeta)}$ and $v_a^{(\zeta)}$, i.e.

$$u_a^{(\zeta)\dagger}u_a^{(\zeta')} = v_a^{(\zeta)\dagger}v_a^{(\zeta')} = \delta_{\zeta\zeta'},\quad u_a^{(\zeta)\dagger}v_a^{(\zeta')} = 0,$$

as well as the method developed in Refs. [2], we find for the zeroth order term $\psi_a^{(0)}(\mathbf{r},t)$ of the expansion (9) the following expression:

$$\psi_a^{(0)}(\mathbf{r},t) = \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}e^{\mathrm{i}\mathbf{p}\mathbf{r}}S_a(\mathbf{p},t)\psi_a(\mathbf{p},0). \tag{10}$$

Here

$$S_a(\mathbf{p},t) = \sum_{\zeta=\pm1}\left[\left(u_a^{(\zeta)}\otimes u_a^{(\zeta)\dagger}\right)\exp\left(-\mathrm{i}E_a^{(\zeta)}t\right) + \left(v_a^{(\zeta)}\otimes v_a^{(\zeta)\dagger}\right)\exp\left(+\mathrm{i}E_a^{(\zeta)}t\right)\right], \tag{11}$$

is the analog of the Pauli-Jordan function for a spin-1/2 particle in an external magnetic field and

$$\psi_a(\mathbf{p},0) = \int\mathrm{d}^3\mathbf{r}e^{-\mathrm{i}\mathbf{p}\mathbf{r}}\psi_a(\mathbf{r},0),$$

is the Fourier transform of the initial wave function (7).

Eqs. (10) and (11) together with Eqs. (3) and (7) [or Eq. (2)] allow one to describe the evolution of any system of two Dirac fermions in an external magnetic field. These expressions directly follow from the Lorentz invariant Dirac-Pauli equation (8), and they are valid for an arbitrary initial condition.

To proceed we now specify our initial condition, that is, give the function $\xi(\mathbf{r})$. Following our previous studies [2], we choose the initial condition for ν_β as a plane wave $\xi(\mathbf{r}) = e^{\mathrm{i}\mathbf{k}\mathbf{r}}\xi_0$, with the initial momentum being directed along the x-axis, i.e. $\mathbf{k} = (k,0,0)$. Given that we have chosen the magnetic field as $\mathbf{B} = (0,0,B)$, we are thus studying the evotution of neutrino wavefunction in a transverse magnetic field. We also consider neutrinos as ultrarelativistic particles, i.e. $k\gg m_{1,2}$, implying $\xi_0^{\mathrm{T}} = (1/2)(1,-1,-1,1)$. It is easy to see that ξ_0 obeys the condition $(1/2)(1-$

$\Sigma_1)\xi_0 = \xi_0$ and that it is normalized to one. Hence, the spinor $\xi(\mathbf{r})$ describes an ultrarelativistic particle propagating along the x-axis with its spin directed opposite to the x-axis, i.e. a left-handedly polarized neutrino.

In contrast to the mixing due to a mass matrix, which is always present, the transition due to a magnetic moment only occurs in the presence of magnetic field. We assume that $B \neq 0$ if $0 \leq t \leq t_0$ and $B = 0$ if $t < 0$ and $t > t_0$. The particle emission and detection are assumed to take place, while the magnetic field is switched off, at $t_e = 0 - \delta t$ and $t_d = t_0 + \delta t$ respectively, where δt is some small period of time. Hence, the operator Σ_1 describes the polarization states of initial and final particles and one can assume that the system is initially prepared to a state described in terms of the spinor ξ_0 as we have done above. Of course, the wave function must be continuous in the borderlines of the $B = 0$ and $B \neq 0$ regions.

Now it is possible to find the field distribution of the particle ν_α initially absent. With help of Eqs. (3), (4), (10) and (11), and by taking into account the initial condition we have chosen, we get for the wavefunction of the right-polarized state of ν_α the following expression:

$$\begin{aligned}\nu_\alpha^{(0)\mathrm{R}}(x,t) =& \frac{1}{2}(1+\Sigma_1)\left[\cos\theta\psi_1^{(0)}(x,t) - \sin\theta\psi_2^{(0)}(x,t)\right] \\ =& \mathrm{i}\sin\theta\cos\theta\left[e^{-\mathrm{i}\mathcal{E}_1 t}\sin(\mu_1 Bt) - e^{-\mathrm{i}\mathcal{E}_2 t}\sin(\mu_2 Bt)\right]e^{\mathrm{i}kx}\kappa_0,\end{aligned} \tag{12}$$

where $\mathcal{E}_{1,2} = \sqrt{m_{1,2}^2 + k^2}$ and $\kappa_0^{\mathrm{T}} = (1/2)(1,1,1,1)$. Here we have neglected the terms proportional to $m_a/k \ll 1$.

The measurable quantity is the square of the wave function $\nu_\alpha^{(0)\mathrm{R}}(x,t)$, which can be regarded as the transition probability for the process $\nu_\beta^{\mathrm{L}} \to \nu_\alpha^{\mathrm{R}}$. From Eq. (12) we obtain for this

$$\begin{aligned}P^{(0)}_{\nu_\beta^{\mathrm{L}}\to\nu_\alpha^{\mathrm{R}}}(t) =& \left|\nu_\alpha^{(0)\mathrm{R}}\right|^2 \\ =& \sin^2(2\theta)\left\{\sin^2(\delta\mu Bt)\cos^2(\bar{\mu}Bt) + \sin(\mu_1 Bt)\sin(\mu_2 Bt)\sin^2\left[\frac{\delta m^2 t}{4k}\right]\right\},\end{aligned} \tag{13}$$

where $\delta m^2 = m_1^2 - m_2^2$, $\delta\mu = (\mu_1 - \mu_2)/2$ and $\bar{\mu} = (\mu_1 + \mu_2)/2$.

In the case of neutrinos having equal magnetic moments, $\mu_1 = \mu_2 = \mu_0$, this leads to the result one would expect. Then Eq. (13) can be rewritten as, $P = P_F P_S$, where $P_F = \sin^2(2\theta)\sin^2[\delta m^2 t/4k]$ is the usual transition probability of flavor oscillations and $P_S = \sin^2(\mu_0 Bt)$ is the probability of transitions between different polarization states within each mass eigenstate. That is, as magnetic moment interactions are in this case insensitive to flavor, the transitions between flavors are solely due to the mass mixing.

Let us now consider the first order correction $\psi_a^{(1)}(\mathbf{r},t)$ to the function $\nu_\alpha^{(0)}$ [see Eq. (9)]. Using the same method as in Ref. [3] one obtains

$$\begin{aligned}\psi_a^{(1)}(\mathbf{r},t) =& -\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}e^{\mathrm{i}\mathbf{pr}}\Big[\left(u_a^{(\zeta)}\otimes u_a^{(\zeta)\dagger}\right)\exp\left(-\mathrm{i}E_a^{(\zeta)}t\right)V\mathcal{G}_a^{(\zeta)} \\ &+ \left(v_a^{(\zeta)}\otimes v_a^{(\zeta)\dagger}\right)\exp\left(+\mathrm{i}E_a^{(\zeta)}t\right)V\mathcal{R}_a^{(\zeta)}\Big]\psi_b(\mathbf{p},0),\end{aligned} \tag{14}$$

where

$$\mathcal{G}_a^{(\zeta)} = \int_0^t \mathrm{d}t\exp\left(+\mathrm{i}E_a^{(\zeta)}t\right)S_b(\mathbf{p},t), \quad \mathcal{R}_a^{(\zeta)} = \int_0^t \mathrm{d}t\exp\left(-\mathrm{i}E_a^{(\zeta)}t\right)S_b(\mathbf{p},t). \tag{15}$$

In Eqs. (14) and (15) $a \neq b$.

With help of Eqs. (3), (4), (14) and (15) we obtain the following first-order correction to $\nu_\alpha^{(0)}$:

$$\nu_\alpha^{(1)\mathrm{R}}(x,t) = \mathrm{i}\cos 2\theta\frac{\mu B}{2}\left(e^{-\mathrm{i}\Sigma t}\frac{\sin\Delta t}{\Delta} + e^{-\mathrm{i}\sigma t}\frac{\sin\delta t}{\delta}\right)e^{\mathrm{i}kx}\kappa_0, \tag{16}$$

where

$$\begin{aligned}\Sigma =& \frac{\mathcal{E}_1 + \mathcal{E}_2}{2} + \bar{\mu}B, \quad \Delta = \frac{\mathcal{E}_1 - \mathcal{E}_2}{2} + \delta\mu B, \\ \sigma =& \frac{\mathcal{E}_1 + \mathcal{E}_2}{2} - \bar{\mu}B, \quad \delta = \frac{\mathcal{E}_1 - \mathcal{E}_2}{2} - \delta\mu B.\end{aligned} \tag{17}$$

It can be seen from Eqs. (16) and (17) that the additional constraint should be fulfilled for the perturbative approach to be valid, namely

$$\frac{\mathcal{E}_1 - \mathcal{E}_2}{2} \neq \pm\delta\mu B. \tag{18}$$

It should be noticed that Eq. (16) is obtained in the ultrarelativistic limit where small terms $m_a/k \ll 1$ are neglected.

Let us now find out the effect of the first-order correction to the transition probability. Using Eqs. (12) and (16) we obtain

$$\begin{aligned} P^{(1)}_{\nu^{\mathrm{L}}_\beta \to \nu^{\mathrm{R}}_\alpha}(t) = &\nu^{(0)\mathrm{R}\dagger}_\alpha \nu^{(1)\mathrm{R}}_\alpha + \text{h.c.} = \frac{\mu B}{4[(\delta m^2/4k)^2 - (\delta\mu B)^2]} \sin 4\theta \Bigg\{ \frac{\delta m^2}{4k} \sin\left(\frac{\delta m^2 t}{2k}\right) \sin(2\delta\mu B t) \\ &- 4\delta\mu B \left[\sin^2(\delta\mu B t)\cos^2(\bar{\mu} B t) + \sin(\mu_1 B t)\sin(\mu_2 B t)\sin^2\left(\frac{\delta m^2 t}{4k}\right)\right]\Bigg\}. \end{aligned} \tag{19}$$

Summarizing, the transition probability of spin-flavor oscillations between mass eigenstate neutrinos up to effects linear in the transition magnetic moment are given as a sum of the probabilities given in Eqs. (13) and (19).

Let us discuss the applicability of our formalism to one of the possible channels of neutrino oscillations, $\nu^{\mathrm{L}}_\mu \xrightarrow{B} \nu^{\mathrm{R}}_\tau$. According to the recent experimental data (see, e.g., Ref. [7]) the mixing angle between ν_μ and ν_τ is close to its maximal value of $\pi/4$. In the limit of maximal mixing the magnetic moment matrix given in Eq. (6) takes the form

$$(\mu_{ab}) \approx \begin{pmatrix} (M_{\tau\tau} + M_{\mu\mu})/2 + M_{\tau\mu} & -(M_{\tau\tau} - M_{\mu\mu})/2 \\ -(M_{\tau\tau} - M_{\mu\mu})/2 & (M_{\tau\tau} + M_{\mu\mu})/2 - M_{\tau\mu} \end{pmatrix}. \tag{20}$$

Our approach is valid in the case this matrix is close to diagonal, i.e. when $|(M_{\tau\tau} - M_{\mu\mu})/2| \ll |(M_{\tau\tau} + M_{\mu\mu})/2 \pm M_{\tau\mu}|$. As is well known, diagonal magnetic moments $M_{\ell\ell}$ are in general very small in the extensions of the Standard Model $M_{\ell\ell} \sim 10^{-19}(m_{\ell\ell}/\mathrm{eV})\mu_{\mathrm{B}}$ (see, e.g., Refs. [8]) while the transition ones, $M_{\tau\mu}$ in our case, can be much bigger (see, e.g., Refs. [9]), close to the experimental upper limit of $10^{-10}\mu_{\mathrm{B}}$ [10]. One can see that for any conceivable values of the masses of the known neutrinos, $M_{\mu\mu}$ and $M_{\tau\tau}$ are orders of magnitude smaller than $10^{-10}\mu_{\mathrm{B}}$, so that a situation where our results are applicable is quite possible.

If the matrix (20) is not close to diagonal, that is, its diagonal elements are much smaller than the non-diagonal ones, the standard quantum mechanical transition probability formula for neutrino spin-flavor oscillations, similar to that derived from the Schrödinger equation in Refs. [1], is to be used for the description of the mass eigenstates evolution. The intermediate case, when the diagonal elements are comparable with the non-diagonal ones, requires separate special attention.

Let us consider the situation $(\mathcal{E}_1 - \mathcal{E}_2)/2 \approx (\delta m^2)/(4k) \sim |\delta\mu B|$. In this case the condition (18) is violated and our results are therefore inapplicable. This is an interesting situation of its own as it may allow for resonant spin-flavor oscillations and enhanced transition probability. This kind of situation could be realized, e.g., in a vicinity of a neutron star where there is a magnetic field of the order $B \sim 10\,\mathrm{MG}$, providing that $\delta m^2 \sim 10^{-5}\,\mathrm{eV}^2$, $k \sim 10\,\mathrm{MeV}$, and $\delta\mu \sim 10^{-10}\mu_{\mathrm{B}}$.

In summary, we have studied in the framework of relativistic quantum mechanics the evolution of a system of two Dirac neutrinos that mix with each other and have non-vanishing magnetic moments. By solving the Dirac-Pauli equation with a given initial condition we determined the time evolution of this system in an external magnetic field. We applied the recently developed approach for the description of the first quantized mixed particles. We obtained the general expressions for the fermion fields distributions exactly accounting for an external magnetic field in the leading and the next-to-leading orders in the transition magnetic moment in the mass eigenstates basis. The derived neutrino wave functions are valid for arbitrary initial conditions since we started with the Lorentz invariant neutrino wave equations. For instance, with help of Eqs. (10) and (14) one can describe the propagation of non-relativistic particles in an external magnetic field. When we considered spin-flavor oscillations of ultrarelativistic neutrinos in a constant transversal magnetic field, we derived the new analytical expression for the transition probability for the spin-flavor conversion in the limit where the energy associated with the magnetic interaction arising from transition magnetic moment is small compared with the total neutrino energy. Then we show that the situation discussed in the present paper is not disfavored by the modern experimental data. Although we consider neutrinos, our formalism is straightforwardly applicable to any spin-1/2 particles.

Acknowledgments

The work has been supported by the Academy of Finland under the contracts No. 108875 and No. 104915. MD is thankful to the Russian Science Support Foundation for a grant as well as to Anatoly Borisov, Andrey Lobanov

(MSU) and Timur Rashba (MPI, Munich) for helpful discussions.

[1] J. Schechter and J. W. F. Valle, Phys. Rev. D **24**, 1883 (1981); M. B. Voloshin, M. I. Vysotskiĭ, and L. B. Okun', JETP **64**, 446 (1986); Sov. J. Nucl. Phys. **44**, 440 (1986); E. Akhmedov, Phys. Lett. B **213**, 64 (1988); C.-S. Lim and W. J. Marciano, Phys. Rev. D **37**, 1368 (1988).

[2] M. Dvornikov, Phys. Lett. B **610**, 262 (2005), hep-ph/0411101; hep-ph/0609139; hep-ph/0610047.

[3] M. Dvornikov, Eur. Phys. J. C **47**, 437 (2006), hep-ph/0601156.

[4] A. Egorov, A. Lobanov, and A. Studenikin, Phys. Lett. B **491**, 137 (2000), hep-ph/9910476; M. S. Dvornikov and A. I. Studenikin, Phys. At. Nucl. **64**, 1624 (2001); *ibid.* **67**, 719 (2004); M. Dvornikov, to be published in Phys. Atom. Nucl., hep-ph/0410152.

[5] A. V. Borisov, V. Ch. Zhukovsky, and A. I. Ternov, Sov. Phys. J. **31**, 228 (1988).

[6] I. M. Ternov, V. G. Bagrov, and A. M. Khapaev, JETP **21**, 613 (1965).

[7] M. Maltoni, T. Schwetz, M. A. Tortola, and J. W. F. Valle, New J. Phys. **6**, 122 (2004), hep-ph/0405172.

[8] K. Fujikawa and R. E. Shrock, Phys. Rev. Lett. **45**, 963 (1980); M. Dvornikov and A. Studenikin, Phys. Rev. D **69**, 073001 (2004), hep-ph/0305206.

[9] J. E. Kim, Phys. Rev. D **14**, 3000 (1976); M. Czakon, J. Gluza, and M. Zralek, Phys. Rev. D **59**, 013010 (1999), hep-ph/9811477.

[10] W.-M. Yao et al., J. Phys. G **33**, 1 (2006).