New approach to the description of neutrino oscillations in various external fields

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Difficulties of the quantum mechanical approach

- Do different neutrino eigenstates have equal energies of equal momenta or equal velocities?
- Do oscillations happen in time or in space?
- Is it necessary to treat neutrinos as spinor particles?
- Should one take into account the coordinate dependence of the neutrino wave function?
- How can one describe oscillations of non-relativistic neutrinos?

Effective Lagrangian for neutrino flavor oscillations in vacuum

- Flavor neutrinos Lagrangian

$$\mathcal{L} = \sum_{\ell=\alpha\beta} \bar{\nu}_\ell (\mathrm{i}\gamma^\mu \partial_\mu - m_\ell)\nu_\ell - \sum_{\substack{\ell,\ell'=\alpha\beta \\ \alpha\neq\beta}} \Delta_{\ell\ell'} \bar{\nu}_\ell \nu_{\ell'}$$

- Initial conditions $\nu_\ell(\mathbf{r}, t=0) = \xi_\ell(\mathbf{r})$

- Fields distributions at subsequent moments of time $\nu_\ell(\mathbf{r}, t) = ?$ at $t > 0$

Evolution of mass eigenstates

- Mass eigenstates $\nu_\ell(\mathbf{r},t) = \sum_a U_{\ell a} \psi_a(\mathbf{r},t)$
- The wave functions of ψ_a satisfying initial conditions

$$\psi_a(\mathbf{r},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \sum_{\zeta=\pm 1} \left\{ \left(u_\zeta \otimes u_\zeta^\dagger \right) e^{-\mathrm{i}E_\zeta t} + \left(v_\zeta \otimes v_\zeta^\dagger \right) e^{\mathrm{i}E_\zeta t} \right\} e^{\mathrm{i}\mathbf{p}\mathbf{r}} \psi_a(\mathbf{p},0)$$

Evolution of flavor neutrinos

- Fields distributions of flavor neutrinos

$$\nu_\ell(\mathbf{r},t) = \sum_{a\ell'} U_{\ell a}(U^{-1})_{a\ell'} \int \mathrm{d}^3\mathbf{r}' S_a(\mathbf{r}'-\mathbf{r},t)(-\mathrm{i}\gamma^0)\xi_{\ell'}(\mathbf{r}')$$

- Pauli-Jordan function

$$S_a(\mathbf{r},t) = (\mathrm{i}\gamma^\mu \partial_\mu + m_a) D_a(\mathbf{r},t),$$

$$D_a(\mathbf{r},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{\mathrm{i}\mathbf{p}\mathbf{r}} \frac{\sin E_a t}{E_a}$$

Evolution of two mixed flavor neutrinos

- Initial conditions $\xi_\beta(\mathbf{r}) = e^{\mathrm{i}\mathbf{kr}}\xi_0,\ \xi_\alpha(\mathbf{r}) = 0$
- We obtain the wave function of ν_α in high initial momentum approximation $(k \gg m_{1,2})$

$$\nu_\alpha(\mathbf{r},t) = e^{\mathrm{i}\mathbf{kr}} \sin 2\theta \sin\left[\Phi(k)t\right]$$

$$\times\left\{\sin\left[\sigma(k)t\right] + \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\cos\left[\sigma(k)t\right]\right\}\xi_0 + \mathcal{O}(m_a / k),$$

$$\Phi(k) \approx \frac{\delta m^2}{4k},\ \sigma(k) \approx k + \frac{m_1^2 + m_2^2}{4k}$$

- Transition probability $P_{\nu_\beta \to \nu_\alpha}(t) = \sin^2(2\theta)\sin^2\left(\frac{\delta m^2}{4k}t\right) + \mathcal{O}(m_a^2 / k^2),$

Effective Lagrangian for neutrino flavor oscillations in matter

- Neutrino interaction with matter is equivalent to the external axial-vector field f^{μ}

$$\mathcal{L} = \sum_{\ell=\alpha\beta} \bar{\nu}_{\ell}(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m_{\ell})\nu_{\ell}$$

$$- \sum_{\substack{\ell,\ell'=\alpha\beta \\ \alpha\neq\beta}} \Delta_{\ell\ell'}\bar{\nu}_{\ell}\nu_{\ell'} - \sum_{\ell=\alpha\beta} \bar{\nu}_{\ell}\gamma_{\mu}^{\mathrm{L}}\nu_{\ell} f_{\ell}^{\mu}$$

Neutrino interaction with moving and polarized matter

- Electron neutrinos can interact with matter by means of both charged and neutral currents; muon or τ-neutrinos – only via neutral currents

$$f_\ell^\mu = \sqrt{2} G_F \sum_{f=e,p,n} \left(j_f^\mu \rho_f^{(\ell)} + \lambda_f^\mu \kappa_f^{(\ell)} \right),$$

$$j_f^\mu = \left(n_f , n_f \mathbf{v}_f \right),$$

$$\lambda_f^\mu = \left(n_f \left(\boldsymbol{\zeta}_f \mathbf{v}_f \right), n_f \boldsymbol{\zeta}_f \sqrt{1 - v_f^2} + \frac{n_f \mathbf{v}_f \left(\boldsymbol{\zeta}_f \mathbf{v}_f \right)}{1 + \sqrt{1 - v_f^2}} \right)$$

Dirac equations for the neutrino mass eigenstates interacting with background matter

$$\mathrm{i}\dot{\psi}_a = \mathcal{H}_a \psi_a + V\psi_b, \ a,b = 1,2, \ a \neq b,$$

$$\mathcal{H}_a = (\boldsymbol{\alpha}\mathbf{p}) + \rho_3 m_a + \rho_3 \gamma_\mu^{\mathrm{L}} g_a^\mu, \ V = \rho_3 \gamma_\mu^{\mathrm{L}} g^\mu,$$

$$g_1^\mu = \cos^2\theta f_\alpha^\mu + \sin^2\theta f_\beta^\mu,$$

$$g_2^\mu = \cos^2\theta f_\beta^\mu + \sin^2\theta f_\alpha^\mu,$$

$$g^\mu = -\sin\theta\cos\theta(f_\alpha^\mu - f_\beta^\mu)$$

- Dirac equations for the neutrino mass eigenstates are coupled!

Energy levels of a neutrino in background matter

$$E_a^{(\zeta)} = \sqrt{\mathbf{p}^2 \left(1 - \zeta \frac{g_a^0}{2|\mathbf{p}|}\right)^2 + m_a^2} + \frac{g_a^0}{2}$$

- See A.Studenikin, A.Ternov, PLB **608** (2005) 107
- We study the case of non-moving and unpolarized matter $(\mathbf{f}_\ell = 0)$
- Here ζ is the neutrino helicity (eigenvalue of $(\mathbf{\Sigma p})/|\mathbf{p}|$)

Basis spinors in background matter

$$u_a^{(\zeta)} = \frac{1}{2}\begin{pmatrix} \phi^+ \alpha_a^+ \\ \zeta\phi^- \alpha_a^+ e^{\mathrm{i}\varphi} \\ \zeta\phi^+ \alpha_a^- \\ \phi^- \alpha_a^- e^{\mathrm{i}\varphi} \end{pmatrix}, \quad v_a^{(\zeta)} = \frac{1}{2}\begin{pmatrix} \phi^+ \beta_a^- \\ \zeta\phi^- \beta_a^- e^{\mathrm{i}\varphi} \\ -\zeta\phi^+ \beta_a^+ \\ -\phi^- \beta_a^+ e^{\mathrm{i}\varphi} \end{pmatrix},$$

$$\alpha_a^{\pm} = \sqrt{1 \pm \frac{m_a}{E_a^{(\zeta)} - g_a^0/2}}, \quad \beta_a^{\pm} = \sqrt{1 \pm \frac{m_a}{E_a^{(\zeta)} + g_a^0/2}},$$

$$\phi_a^{\pm} = \sqrt{1 \pm \zeta p_3/|\mathbf{p}|}, \quad \tan\varphi = p_2/p_1$$

Transition probability for flavor oscillations

- Ultrarelativistic neutrinos $\boldsymbol{k \gg m_a}$
- Low density matter $\boldsymbol{G_{\mathrm{F}} n_{\mathrm{eff}} \ll k}$

$$P_{\nu_\beta \to \nu_\alpha}(t) = \sin^2(2\theta_{\mathrm{eff}}) \sin^2\left(\frac{\pi t}{L_{\mathrm{eff}}}\right) + \mathcal{O}(m_a^2 / k^2),$$

$$\sin^2(2\theta_{\mathrm{eff}}) = \frac{\Phi^2(k)\sin^2(2\theta)}{\left[\Phi(k)\cos 2\theta - A/2\right]^2 + \Phi^2(k)\sin^2(2\theta)},$$

$$\frac{\pi}{L_{\mathrm{eff}}} = \sqrt{\left[\Phi(k)\cos 2\theta - A/2\right]^2 + \Phi^2(k)\sin^2(2\theta)}, \quad A = f_\beta^0 - f_\alpha^0$$

Effective Lagrangian for neutrino spin-flavor oscillations in electromagnetic fields

- The Lagrangian for the ***Dirac*** neutrinos system interacting with an external magnetic field by means of the magnetic moments is

$$\mathcal{L} = \sum_{\ell=\alpha\beta} \bar{\nu}_\ell (\mathrm{i}\gamma^\mu \partial_\mu - m_\ell)\nu_\ell$$

$$- \sum_{\substack{\ell,\ell'=\alpha\beta \\ \alpha\neq\beta}} \Delta_{\ell\ell'} \bar{\nu}_\ell \nu_{\ell'} - \frac{1}{2} \sum_{\ell,\ell'=\alpha\beta} M_{\ell\ell'} \bar{\nu}_\ell \sigma_{\mu\nu} \nu_{\ell'} F^{\mu\nu}$$

Dirac equations for the neutrino mass eigenstates in the external magnetic field

$$\mathrm{i}\dot{\psi}_a = \mathcal{H}_a \psi_a + \mathrm{V}\psi_b,$$

$$\mathcal{H}_a = (\boldsymbol{\alpha}\mathbf{p}) + \rho_3 m_a - \mu_a \rho_3 \Sigma_3 B,$$

$$\mathrm{V} = -\mu\rho_3\Sigma_3 B, \ \left(\mu_{ab}\right) = \sum_{\ell\ell'=\alpha,\beta} U^{-1}_{a\ell} M_{\ell\ell'} U_{\ell' b}$$

$$\mathbf{B} = (0, 0, B), \ \mathbf{E} = 0$$

- Dirac equations for the neutrino mass eigenstates are coupled!

Energy levels for a neutrino in the magnetic field

$$E_a^{(\zeta)} = \sqrt{p_3^2 + \mathcal{E}_a^{(\zeta)2}}, \; \mathcal{E}_a^{(\zeta)} = \mathcal{E}_a - \zeta\mu_a B,$$

$$\mathcal{E}_a = \sqrt{m_a^2 + p_1^2 + p_2^2}$$

- See I.M.Ternov, *et al.*, JETP **21** (1965) 613
- The quantum number $\zeta=\pm 1$ characterizes the spin direction with respect to the magnetic field (it is the eigenvalue of the operator $\mathbf{\Pi}_a = m_a \Sigma_3 + \rho_2 [\mathbf{\Sigma}\times\mathbf{p}]_3 - \mu_a B$)

Basis spinors in the magnetic field

$$u_a^{(\zeta)} = \frac{1}{2\sqrt{E_a^{(\zeta)}}} \begin{pmatrix} \phi_a^+ \alpha_a^+ \\ -\zeta \phi_a^- \alpha_a^- e^{\mathrm{i}\varphi} \\ \phi_a^+ \alpha_a^- \\ \zeta \phi_a^- \alpha_a^+ e^{\mathrm{i}\varphi} \end{pmatrix}, \quad v_a^{(\zeta)} = \frac{1}{2\sqrt{E_a^{(\zeta)}}} \begin{pmatrix} \phi_a^+ \alpha_a^- \\ \zeta \phi_a^- \alpha_a^- e^{\mathrm{i}\varphi} \\ -\phi_a^+ \alpha_a^- \\ \zeta \phi_a^- \alpha_a^- e^{\mathrm{i}\varphi} \end{pmatrix},$$

$$\phi_a^\pm = \sqrt{1 \pm \zeta m_a / \mathcal{E}_a}, \quad \alpha_a^\pm = \sqrt{E_a^{(\zeta)} \pm \zeta \mathcal{E}_a^{(\zeta)}}, \quad \tan\varphi = p_2 / p_1$$

Initial conditions for spin-flavor oscillations

- We suppose that only left-handed neutrinos of one flavor are presented initially

$$\nu_\beta^{\mathrm{L}}(\mathbf{r},0) = e^{i\mathbf{kr}}\xi_0, \quad \nu_\beta^{\mathrm{R}}(\mathbf{r},0) = 0,$$

$$\nu_\alpha^{\mathrm{L}}(\mathbf{r},0) = 0, \quad \nu_\alpha^{\mathrm{R}}(\mathbf{r},0) = 0$$

Final wave function of ν_α^{R}

- Ultrarelativistic neutrinos $k \gg m_a$

$$\nu_\alpha^{\mathrm{R}}(x,t) = \left\{ \frac{\sin\theta\cos\theta}{2\mathrm{i}} \right.$$

$$\times \left[\frac{\omega_+}{\Omega_+} \sin(\Omega_+ t) \exp(\mathrm{i}\bar{\mu}Bt) - \frac{\omega_-}{\Omega_-} \sin(\Omega_- t) \exp(-\mathrm{i}\bar{\mu}Bt) \right]$$

$$\left. + i\mu B \left[\frac{\sin(\Omega_+ t)}{\Omega_+} \cos^2\theta - \frac{\sin(\Omega_- t)}{\Omega_-} \sin^2\theta \right] \cos(\bar{\mu}Bt) \right\}$$

$$\times \exp(-\mathrm{i}\bar{\mathcal{E}}t + \mathrm{i}kx)\kappa_0 + \mathcal{O}(m_a / k)$$

Spin-flavor oscillations for different magnetic moments matrices

- "Dirac" type magnetic moments, $\boldsymbol{\mu_a \gg \mu}$

$$P_{\nu_\beta^{\mathrm{L}} \to \nu_\alpha^{\mathrm{R}}}(t) = \sin^2(2\theta)\{\sin^2(\delta\mu B t)\cos^2(\bar{\mu} B t) \qquad \delta\mu = (\mu_1 - \mu_2)/2,$$

$$+\sin(\mu_1 B t)\sin(\mu_2 B t)\sin^2[\Phi(k) t]\}, \qquad \bar{\mu} = (\mu_1 + \mu_2)/2$$

- "Majorana" type magnetic moments, $\boldsymbol{\mu \gg \mu_a}$

$$P_{\nu_\beta^{\mathrm{L}} \to \nu_\alpha^{\mathrm{R}}}(t) = \cos^2(2\theta)\left(\frac{\mu B}{\Omega}\right)^2 \sin^2(\Omega t), \ \Omega = \sqrt{(\mu B)^2 + \Phi^2(k)}$$

- Oscillations between $\boldsymbol{\nu_\mu}$ and $\boldsymbol{\nu_\tau}$ $(\boldsymbol{\theta \approx \pi/4})$ in the minimally extended standard model, $\boldsymbol{\mu_1 = \mu_2}$

$$P_{\nu_\beta^{\mathrm{L}} \to \nu_\alpha^{\mathrm{R}}}(t) = \left[\frac{\Phi(k)}{\Omega}\right]^2 \sin^2(\Omega t)\sin^2(\bar{\mu} B t)$$

Discussion

- We exactly solved initial condition problem for the flavor neutrinos system with mixing in vacuum, in an external axial-vector field (interaction with matter) and in an external electromagnetic field. The classical field theory approach was used.
- The leading terms in transition probability expressions for oscillations in vacuum and in matter reproduce the previous results.
- We received the new transition probabilities for ***Dirac*** neutrinos interacting with an external magnetic field.
- It was shown that neutrinos oscillate in time in frames of this approach.
- We obtained small (suppressed by the factor $m_a/k \ll 1$) rapidly oscillating corrections (on the frequency $\omega_{\mathrm{rapid}} \approx k$) to the transition probability formulas.