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Evolution of coupled fermions under the influence of an external axial-vector field

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Abstract. The evolution of coupled fermions interacting with external axial-vector fields is described with the help of the classical field theory. We formulate the initial conditions problem for the system of two coupled fermions in $(3+1)$-dimensional space-time. This problem is solved using perturbation theory. We obtain in the explicit form the expressions for the leading and next to leading order terms in the expansion over the strength of the external fields. It is shown that in the relativistic limit the intensity of the fermion field coincides with the transition probability in the two neutrinos system interacting with moving and polarized matter.

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1 Introduction

The description of mixed fermion evolution has attracted considerable attention since the experimental confirmation of solar neutrino oscillations [1, 2]. The majority of the neutrino oscillations studies involve the quantum mechanical approach to the description of the neutrino wave function evolution [3]. Despite the fact that quantum mechanics allows one to establish the main properties of the neutrino oscillations process, this method of treatment of neutrino oscillations has several disadvantages. Neutrinos are usually supposed to be scalar particles without reference to the multicomponent single neutrino wave function. The famous Pontecorvo formula [4], which is in use in many theoretical and experimental studies of neutrino oscillations, is valid only for ultrarelativistic particles. However, the exact theory of the considered process must be applicable for neutrinos with arbitrary energies. A quantum mechanical approach also does not make it clear if mass or flavor eigenstates bear more physical meaning. Therefore one can see that a theoretical model of neutrino oscillations, which would overcome the above mentioned difficulties, should be put forward. There have been numerous attempts to construct the appropriate formalism for neutrino flavor oscillations in vacuum. The quantum field theory was applied to this problem in [5–8]. The authors of these papers reproduced the Pontecorvo formula and discussed the corrections to this expression. Recently we revealed in [9] that neutrino flavor oscillations in vacuum could be explained in the framework of classical field theory.

It was also realized that neutrino interactions with external fields can drastically change the picture of the oscillations process. For example, it was discovered in [10, 11] that the transition probability can achieve large values if a neutrino interacts with background matter by means of weak currents. Thus we should develop now not only the appropriate theory of neutrino oscillations in vacuum, but also include in our treatment possible effects of neutrino interactions with external fields. During the last three decades the approaches to the theoretical substantiation of the Mikheyev–Smirnov–Wolfenstein (MSW) effect have been developed. Among them we can distinguish [12, 13] in which the neutrino relativistic wave equations in dense matter were analyzed. The S-matrix approach was used [14] to account for the MSW effect. The influence of moving and polarized matter was described in [15, 16].

The purpose of the present work is to provide a deeper understanding of the neutrino flavor oscillations phenomenon. The approach developed in our paper can not only reproduce the Pontecorvo formula for the transition probability, but also give a clear physical explanation for the corrections to this expression, which are widely discussed now (see, e.g., [17] and references therein). The analysis used in this article is based on classical field theory methods. We study the evolution of coupled classical fermions under the influence of external axial-vector fields. A *classical fermion* is regarded as a first quantized field because the Dirac equation,

$$\frac{1}{c}\frac{\partial\psi}{\partial t}+\boldsymbol{\alpha}\boldsymbol{\nabla}\psi+\frac{\mathrm{i}mc}{\hbar}\beta\psi=0\,,$$

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in which we use the common notation for the gamma matrices $\boldsymbol{\alpha} = \gamma^0\gamma$ and $\beta = \gamma^0$, does contain the Plank constant $\hbar$; however, the wave function ψ is supposed to be a non-operator object in our approach. Therefore we do not involve the second quantization in the present work. Note that the discussion of the first quantized neutrino fields is also presented in [18].

In Sect. 2 we start from the flavor neutrino Lagrangian which accounts for the interaction with external axial-vector fields. Then we derive the basic integro-differential equations for the "mass eigenstates" which exactly take into account both Lorentz invariance and the interaction with external fields. These equations are also valid in (3+1) dimensions. The perturbation theory is used for the analysis of the equations obtained. In Sect. 3 we discuss the neutrino fields distributions and obtain the zero order term in their expansion over the external fields strength. This result corresponds to vacuum neutrino oscillations. In Sect. 4 we get the first order correction to the neutrino field intensity in vacuum and show that our formula is identical to the expression for the neutrino transition probability obtained in the quantum mechanical approach for ultrarelativistic neutrinos. This case corresponds to neutrino flavor oscillations in moving and polarized matter. Finally we discuss our results in Sect. 5.

2 General formalism

Without losing generality we can discuss the evolution of the two coupled fermions system (ν_1, ν_2). These fermions are taken to interact with the external axial-vector fields $f^\mu_{1,2}$. The Lagrangian for this system is expressed in the following form:

$$\mathcal{L}(\nu_1,\nu_2) = \sum_{k=1,2} \mathcal{L}_0(\nu_k) + g\bar{\nu}_2\nu_1 + g^*\bar{\nu}_1\nu_2 - \sum_{k=1,2} \bar{\nu}_k \gamma^L_\mu \nu_k f^\mu_k \, , \qquad (1)$$

where g is the coupling constant (g^* is the complex conjugate value), $\gamma^L_\mu = \gamma_\mu(1+\gamma^5)/2$ ($\gamma^5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$), and

$$\mathcal{L}_0(\nu_k) = \bar{\nu}_k(\mathrm{i}\gamma^\mu\partial_\mu - \mathfrak{m}_k)\nu_k \, .$$

is the free fermion Lagrangian, $\mathfrak{m}_k$ are the masses of the corresponding fermions ν_k.

One of the possible examples of the fermions ν_k is the system of neutrinos belonging to different flavor states. In this case we can identify the first fermion in (1) with a muon neutrino ν_μ or a τ-neutrino ν_τ and the second one with an electron neutrino ν_e. These neutrino types are known to interact with matter composed of electrons, protons and neutrons by means of the electroweak interaction. Note that an electron neutrino interacts with background fermions via both charged and neutral weak currents whereas a muon or a τ-neutrino is involved only in the interaction through weak neutral currents. Thus the external axial-vector fields $f^\mu_{1,2}$ can be expressed in terms of the hydrodynamical currents j^μ_f and the polarizations λ^μ_f of different fermions in matter (see, e.g., [16, 19, 20]),

$$f^\mu_{1,2} = \sqrt{2}G_\mathrm{F} \sum_{f=e,p,n} \left(j^\mu_f \rho^{(1,2)}_f + \lambda^\mu_f \kappa^{(1,2)}_f \right) , \qquad (2)$$

where G_F is the Fermi constant and

$$\begin{aligned} \rho^{(1)}_f &= \left(I^{(f)}_{3L} - 2Q^{(f)}\sin^2\theta_\mathrm{W} + \delta_{ef} \right) , \\ \kappa^{(1)}_f &= -\left(I^{(f)}_{3L} + \delta_{ef} \right) , \\ \rho^{(2)}_f &= \left(I^{(f)}_{3L} - 2Q^{(f)}\sin^2\theta_\mathrm{W} \right) , \\ \kappa^{(2)}_f &= -I^{(f)}_{3L} , \\ \delta_{ef} &= \begin{cases} 1, & f = e\,; \\ 0, & f = n, p\,. \end{cases} \end{aligned} \qquad (3)$$

Here $I^{(f)}_{3L}$ is the third isospin component of the matter fermion f, $Q^{(f)}$ is its electric charge and θ_W is the Weinberg angle. The hydrodynamical currents and the polarizations are related to the fermion velocities $\mathbf{v}_f$ and the spin vectors $\boldsymbol{\zeta}_f$ by means of the following formulas:

$$\begin{aligned} j^\mu_f &= (n_f, n_f\mathbf{v}_f) , \\ \lambda^\mu_f &= \left(n_f(\boldsymbol{\zeta}_f\mathbf{v}_f), n_f\boldsymbol{\zeta}_f\sqrt{1-v_f^2} + \frac{n_f\mathbf{v}_f(\boldsymbol{\zeta}_f\mathbf{v}_f)}{1+\sqrt{1-v_f^2}} \right) . \end{aligned} \qquad (4)$$

The detailed derivation of (2)–(4) is presented in [16, 19, 20].

Following the results of our previous work [9] we will study the evolution of the system (1) by solving the Cauchy problem. Let us choose the initial conditions in the form

$$\nu_1(\mathbf{r},0) = 0, \quad \nu_2(\mathbf{r},0) = \xi(\mathbf{r}) , \qquad (5)$$

where $\xi(\mathbf{r})$ is a known function. If one considers the fermions ν_k as flavor neutrinos, the initial conditions in (5) correspond to the common situation in a neutrino oscillation experiment, i.e. $\nu_{\mu,\tau}$ are absent initially and ν_e has some known field distribution. We will be interested in searching for the fields distributions $\nu_k(\mathbf{r},t)$ for $t > 0$.

In order to solve the Cauchy problem we introduce the new set of fermions (ψ_1, ψ_2) by means of the matrix transformation

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \qquad (6)$$

The mixing matrix (which is parameterized with the help of the one angle θ) in (6) is chosen in such a way as to eliminate the second and the third terms in (1). If we had studied the evolution of our system without the external fields $f^\mu_{1,2}$, i.e. in vacuum, the fermions ψ_k would have been called mass eigenstates because they would have diagonalized the Lagrangian and therefore ψ_k would have had definite masses. In our case ($f^\mu_{1,2} \neq 0$), if we simplify the vacuum mixing terms, the matter term becomes more complicated compared to (1). The Lagrangian rewritten in terms of the fields ψ_k can be expressed in the following way:

$$\begin{aligned}\mathcal{L}(\psi_1,\psi_2) = \sum_{k=1,2} \mathcal{L}_0(\psi_k) - \Big[& \bar{\psi}_1 \gamma_\mu^L \psi_1 \left(c^2 f_1^\mu + s^2 f_2^\mu\right) \\ & + \bar{\psi}_2 \gamma_\mu^L \psi_2 \left(c^2 f_2^\mu + s^2 f_1^\mu\right) \\ & + sc \left(\bar{\psi}_1 \gamma_\mu^L \psi_2 + \bar{\psi}_2 \gamma_\mu^L \psi_1\right) \left(f_1^\mu - f_2^\mu\right) \Big] , \end{aligned} \quad (7)$$

where

$$\mathcal{L}_0(\psi_k) = \bar{\psi}_k \left(\mathrm{i}\gamma^\mu \partial_\mu - m_k\right) \psi_k \, .$$

It should be noted that the masses m_k of the fermions ψ_k are related to the masses $\mathfrak{m}_k$ by the formula,

$$m_1 = c^2 \mathfrak{m}_1 + s^2 \mathfrak{m}_2 \, , \quad m_2 = c^2 \mathfrak{m}_2 + s^2 \mathfrak{m}_1 \, ,$$

In (7) we use the notation $c = \cos\theta$ and $s = \sin\theta$.

Equation (7) has some advantages in comparison with (1) despite the more complicated matter interaction term. The terms $g\bar{\nu}_2\nu_1$ and $g^*\bar{\nu}_1\nu_2$ in (1), which are responsible for the vacuum oscillations cannot be treated within the perturbation theory. In order to describe the flavor changing processes one has to take into account these terms exactly. In (7) we have the additional interaction terms which can be analyzed in the usual way with the help of the perturbation theory if the strength of the fields $f_{1,2}^\mu$ is supposed to be weak. The criterion of the weakness of the external fields is discussed in detail in Sect. 4. It is also possible to assume the convenient time dependence (sometimes it is necessary to consider the adiabatic switching-on of the interaction) of the fields $f_{1,2}^\mu$ in (7). On the contrary we cannot “switch-on” or “switch-off” the constant g in (1) at some moments of time because it is related to the properties of a theoretical scheme underlying the phenomenological model of neutrino oscillations.

Now let us discuss the evolution of the system (7) with the initial conditions

$$\psi_1(\mathbf{r},0) = -s\xi(\mathbf{r}) \, , \quad \psi_2(\mathbf{r},0) = c\xi(\mathbf{r}) \, ,$$

which follow from (5) and (6). We also suppose that the external fields $f_{1,2}^\mu$ are weak and one is able to take them into account in the lowest order of perturbation theory. If we had solved this problem using quantum field theory, we would have calculated the contributions of the four Feynman diagrams shown on Fig. 1.

We can always rewrite the Dirac equations which result from (7) in the form

$$\begin{aligned} \mathrm{i}\dot{\psi}_1 &= (H_1 + V_1)\psi_1 + V\psi_2 \, , \\ \mathrm{i}\dot{\psi}_2 &= (H_2 + V_2)\psi_2 + V\psi_1 \, , \end{aligned} \quad (8)$$

where $V_1 = \beta\gamma_\mu^L (c^2 f_1^\mu + s^2 f_2^\mu)$, $V_2 = \beta\gamma_\mu^L (c^2 f_2^\mu + s^2 f_1^\mu)$, $V = sc\beta\gamma_\mu^L (f_1^\mu - f_2^\mu)$ and $H_k = -\mathrm{i}\boldsymbol{\alpha}\nabla + \beta m_k$ are the free fields Hamiltonians. We are searching for the solutions of (8) in the following way:

$$\psi_k(\mathbf{r},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}} \left[a_k(\mathbf{p},t) \Psi_{k,\mathbf{p}}^+(x) + b_k(\mathbf{p},t) \Psi_{k,\mathbf{p}}^-(x) \right] , \quad (9)$$

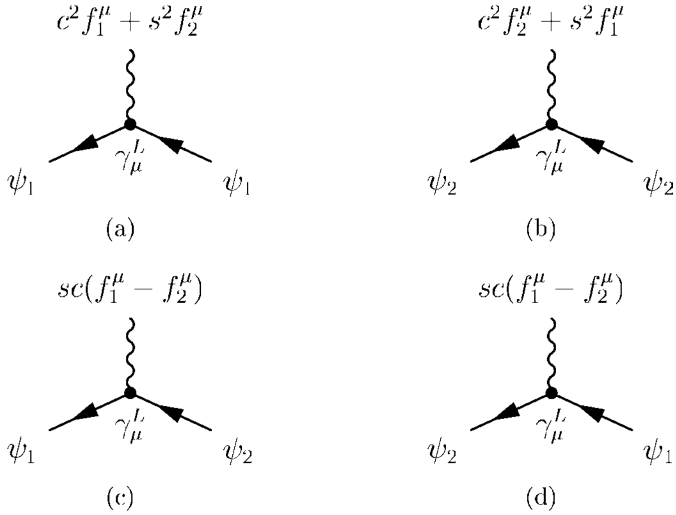

Fig. 1. Feynman diagrams contributing to the interaction of the fermions ψ_k with the external axial-vector fields $f_{1,2}^\mu$

where $\Psi_{k,\mathbf{p}}^+(x) = u_k(\mathbf{p})\mathrm{e}^{-\mathrm{i}p_k x}$ and $\Psi_{k,\mathbf{p}}^-(x) = v_k(\mathbf{p})\mathrm{e}^{\mathrm{i}p_k x}$ are the basis spinors, $x^\mu = (t,\mathbf{r})$, $p_k^\mu = (\mathcal{E}_k,\mathbf{p})$ and $\mathcal{E}_k = \sqrt{\mathbf{p}^2 + m_k^2}$ is the energy of the fermion ψ_k. The coefficients $a_k(\mathbf{p},t)$ and $b_k(\mathbf{p},t)$ are not the creation and annihilation operators since we are using here the classical field theory. The values of these functions should be chosen in such a way as to satisfy the initial conditions (5). Note that there is an additional time dependence of these functions in contrast to the case of the flavor changing process in vacuum [9].

Using the orthonormality condition of the basis spinors $\Psi_{k,\mathbf{p}}^+(x) = u_k(\mathbf{p})\mathrm{e}^{-\mathrm{i}p_k x}$ and $\Psi_{k,\mathbf{p}}^-(x) = v_k(\mathbf{p})\mathrm{e}^{\mathrm{i}p_k x}$ we obtain from (8) the new integro-differential equations for the functions $a_k(\mathbf{p},t)$ and $b_k(\mathbf{p},t)$,

$$\begin{aligned} \mathrm{i}\dot{a}_1(\mathbf{p},t) &= \frac{1}{(2\pi)^{3/2}} \bigg(\int \mathrm{d}^3\mathbf{r} \Psi_{1,\mathbf{p}}^{+\dagger}(x) V_1 \psi_1(\mathbf{r},t) \\ &\quad + \int \mathrm{d}^3\mathbf{r} \Psi_{1,\mathbf{p}}^{+\dagger}(x) V \psi_2(\mathbf{r},t) \bigg) , \\ \mathrm{i}\dot{b}_1(\mathbf{p},t) &= \frac{1}{(2\pi)^{3/2}} \bigg(\int \mathrm{d}^3\mathbf{r} \Psi_{1,\mathbf{p}}^{-\dagger}(x) V_1 \psi_1(\mathbf{r},t) \\ &\quad + \int \mathrm{d}^3\mathbf{r} \Psi_{1,\mathbf{p}}^{-\dagger}(x) V \psi_2(\mathbf{r},t) \bigg) , \\ \mathrm{i}\dot{a}_2(\mathbf{p},t) &= \frac{1}{(2\pi)^{3/2}} \bigg(\int \mathrm{d}^3\mathbf{r} \Psi_{2,\mathbf{p}}^{+\dagger}(x) V_2 \psi_2(\mathbf{r},t) \\ &\quad + \int \mathrm{d}^3\mathbf{r} \Psi_{2,\mathbf{p}}^{+\dagger}(x) V \psi_1(\mathbf{r},t) \bigg) , \\ \mathrm{i}\dot{b}_2(\mathbf{p},t) &= \frac{1}{(2\pi)^{3/2}} \bigg(\int \mathrm{d}^3\mathbf{r} \Psi_{2,\mathbf{p}}^{-\dagger}(x) V_2 \psi_2(\mathbf{r},t) \\ &\quad + \int \mathrm{d}^3\mathbf{r} \Psi_{2,\mathbf{p}}^{-\dagger}(x) V \psi_1(\mathbf{r},t) \bigg) . \end{aligned} \quad (10)$$

These equations correctly take into account both Lorentz invariance and the interaction with the external fields $f_{1,2}^\mu$. However if we suppose that these external fields are rather weak, it is possible to look for the solutions of (10) in the form of a series,

$$\begin{aligned} a_k(\mathbf{p},t) =& a_k^{(0)}(\mathbf{p}) + a_k^{(1)}(\mathbf{p},t) + \dots, \\ b_k(\mathbf{p},t) =& b_k^{(0)}(\mathbf{p}) + b_k^{(1)}(\mathbf{p},t) + \dots. \end{aligned} \tag{11}$$

Equations (11) mean that the field distributions can also be presented in the form of the series $\psi_k(\mathbf{r},t) = \psi_k^{(0)}(\mathbf{r},t) + \psi_k^{(1)}(\mathbf{r},t) + \dots$ Note that the coefficients $a_k^{(0)}(\mathbf{p})$ and $b_k^{(0)}(\mathbf{p})$, which correspond to the function $\psi_k^{(0)}(\mathbf{r},t)$, do not depend on time. The functions $\psi_k^{(0)}(\mathbf{r},t)$ are responsible for the evolution of the considered system in vacuum, i.e. at $f_{1,2}^{\mu} = 0$. These functions in 2-dimensional space-time have been found in our previous work [9] where the analogous Cauchy problem has been solved in explicit form in the $(1+1)$-dimensional case. However to describe the evolution of our system with the non-zero external fields in $(3+1)$-dimensional space-time we should study the vacuum case also in $(3+1)$ dimensions.

3 Evolution of the system in vacuum

To study the behavior of $\psi_k^{(0)}(\mathbf{r},t)$, i.e. the evolution of our system in vacuum, we use the results of [9] where it has been revealed that the field distributions for the given initial conditions have the form

$$\begin{aligned} \psi_1^{(0)}(\mathbf{r},t) &= -s\int \mathrm{d}^3\mathbf{r}' S_1(\mathbf{r}'-\mathbf{r},t)(-\mathrm{i}\beta)\xi(\mathbf{r}'), \\ \psi_2^{(0)}(\mathbf{r},t) &= c\int \mathrm{d}^3\mathbf{r}' S_2(\mathbf{r}'-\mathbf{r},t)(-\mathrm{i}\beta)\xi(\mathbf{r}'), \end{aligned} \tag{12}$$

where $S_k(\mathbf{r},t)$ is the Pauli–Jordan function for the fermion ψ_k (see, e.g., [21]). It should be noted that (12) are the most general ones and valid in $(3+1)$ dimensions. Contrary to the approach of [9] here we use the momentum representation because the integrations in (12) are rather cumbersome in the coordinate representation. Thus one rewrites these expressions using the Fourier transform of the initial conditions. Equations (12) now take the form

$$\begin{aligned} \psi_1^{(0)}(\mathbf{r},t) &= -s\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\mathbf{pr}} S_1(-\mathbf{p},t)(-\mathrm{i}\beta)\xi(\mathbf{p}), \\ \psi_2^{(0)}(\mathbf{r},t) &= c\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\mathbf{pr}} S_2(-\mathbf{p},t)(-\mathrm{i}\beta)\xi(\mathbf{p}), \end{aligned} \tag{13}$$

where

$$S_k(-\mathbf{p},t) = \left[\cos\mathcal{E}_k t - i\frac{\sin\mathcal{E}_k t}{\mathcal{E}_k}(\boldsymbol{\alpha}\mathbf{p} + \beta m_k)\right](\mathrm{i}\beta), \tag{14}$$

and

$$\xi(\mathbf{p}) = \int \mathrm{d}^3\mathbf{r}\mathrm{e}^{-\mathrm{i}\mathbf{pr}}\xi(\mathbf{r}),$$

are the Fourier transforms of the functions $S_k(\mathbf{r},t)$ and $\xi(\mathbf{r})$.

Now let us choose the initial condition. We suppose that the initial field distribution of ν_2 is the plane wave, i.e. $\xi(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}\xi_0$, where ξ_0 is the normalization spinor. The Fourier transform of this function can be simply computed, $\xi(\mathbf{p}) = (2\pi)^3\delta^3(\omega - \mathbf{p})\xi_0$. Using (13) we get the field distributions in the $(3+1)$-dimensional space-time for the plane wave initial condition,

$$\begin{aligned} \psi_1^{(0)}(\mathbf{r},t) &= -s\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}\left[\cos[\mathcal{E}_1(\omega)t] - \mathrm{i}\frac{\sin[\mathcal{E}_1(\omega)t]}{\mathcal{E}_1(\omega)}(\boldsymbol{\alpha\omega} + \beta m_1)\right] \\ &\quad \times \xi_0, \\ \psi_2^{(0)}(\mathbf{r},t) &= c\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}\left[\cos[\mathcal{E}_2(\omega)t] - \mathrm{i}\frac{\sin[\mathcal{E}_2(\omega)t]}{\mathcal{E}_2(\omega)}(\boldsymbol{\alpha\omega} + \beta m_2)\right] \\ &\quad \times \xi_0, \end{aligned} \tag{15}$$

where $\mathcal{E}_k(\omega) = \sqrt{\omega^2 + m_k^2}$.

In the following we discuss the case of rapidly oscillating initial conditions, i.e. $\omega \gg m_{1,2}$. One obtains from (15) the field distributions for $\omega \gg m_{1,2}$ in the following form:

$$\begin{aligned} \psi_1^{(0)}(\mathbf{r},t) &= -s\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}(\cos[\mathcal{E}_1(\omega)t] - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\mathcal{E}_1(\omega)t])\xi_0, \\ \psi_2^{(0)}(\mathbf{r},t) &= c\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}(\cos[\mathcal{E}_2(\omega)t] - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\mathcal{E}_2(\omega)t])\xi_0, \end{aligned} \tag{16}$$

where $\mathbf{n} = \boldsymbol{\omega}/\omega$ is the unit vector in the direction of the initial field distribution momentum. The fermion ν_1 is absent at $t = 0$. Therefore it would be interesting to examine the field distribution $\nu_1^{(0)}(\mathbf{r},t)$ at $t > 0$. Using (6) and (16), we obtain

$$\begin{aligned} \nu_1^{(0)}(\mathbf{r},t) = c\psi_1^{(0)} + s\psi_2^{(0)} &= \sin 2\theta \sin[\Delta(\omega)t] \\ &\times \left\{\sin[\sigma(\omega)t] + \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\cos[\sigma(\omega)t]\right\}\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}\xi_0, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \sigma(\omega) &= \frac{\mathcal{E}_1(\omega) + \mathcal{E}_2(\omega)}{2} \to \omega + \frac{m_1^2 + m_2^2}{4\omega}, \\ \Delta(\omega) &= \frac{\mathcal{E}_1(\omega) - \mathcal{E}_2(\omega)}{2} \to \frac{m_1^2 - m_2^2}{4\omega} = \frac{\Delta m^2}{4\omega}. \end{aligned}$$

The measurable quantity of a classical spinor field is the intensity. With the help of (16) one gets the intensity of the fermion $\nu_1^{(0)}$ in the following form:

$$\begin{aligned} I_1^{(0)}(t) = |\nu_1^{(0)}(\mathbf{r},t)|^2 &= \sin^2(2\theta)\sin^2[\Delta(\omega)t]\xi_0^\dagger|\sin[\sigma(\omega)t] \\ &\quad + \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\cos[\sigma(\omega)t]|^2\xi_0 \\ &= \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4\omega}t\right), \end{aligned} \tag{18}$$

which reproduces the Pontecorvo formula in the $(3+1)$-dimensional space-time since we can regard the intensity of the fermion ν_1 as the transition probability in two neutrino system. Equation (18) also generalizes the result of our previous work [9] where the analogous expression was derived in $(1+1)$ dimensions.

4 Interaction of the system with an external field

In order to proceed in our study of the two neutrino system evolution under the influence of the external fields $f^{\mu}_{1,2}$ we discuss the further correction to the vacuum case. The first order corrections to (13) can be derived from (9) and (10) and have the form

$$\begin{aligned}
\psi_1^{(1)}(\mathbf{r},t) = {} & \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{pr}}}{2\mathcal{E}_1}\Big\{\mathcal{E}_1\Big[sV_1\left(\mathfrak{S}_1^{+}\mathrm{e}^{-\mathrm{i}\mathcal{E}_1 t}+\mathfrak{S}_1^{-}\mathrm{e}^{+\mathrm{i}\mathcal{E}_1 t}\right) \\
& -cV(\mathfrak{S}_{12}^{+}\mathrm{e}^{-\mathrm{i}\mathcal{E}_1 t}+\mathfrak{S}_{12}^{-}\mathrm{e}^{+\mathrm{i}\mathcal{E}_1 t})\Big] \\
& +(\boldsymbol{\alpha}\mathbf{p}+\beta m_1)\Big[sV_1\left(\mathfrak{S}_1^{+}\mathrm{e}^{-\mathrm{i}\mathcal{E}_1 t}-\mathfrak{S}_1^{-}\mathrm{e}^{+\mathrm{i}\mathcal{E}_1 t}\right) \\
& -cV\left(\mathfrak{S}_{12}^{+}\mathrm{e}^{-\mathrm{i}\mathcal{E}_1 t}-\mathfrak{S}_{12}^{-}\mathrm{e}^{+\mathrm{i}\mathcal{E}_1 t}\right)\Big]\Big\}(-\mathrm{i}\beta)\xi(\mathbf{p})\,, \\
\psi_2^{(1)}(\mathbf{r},t) = {} & -\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{pr}}}{2\mathcal{E}_2}\Big\{\mathcal{E}_2\Big[cV_2\left(\mathfrak{S}_2^{+}\mathrm{e}^{-\mathrm{i}\mathcal{E}_2 t}+\mathfrak{S}_2^{-}\mathrm{e}^{+\mathrm{i}\mathcal{E}_2 t}\right) \\
& -sV\left(\mathfrak{S}_{21}^{+}\mathrm{e}^{-\mathrm{i}\mathcal{E}_2 t}+\mathfrak{S}_{21}^{-}\mathrm{e}^{+\mathrm{i}\mathcal{E}_2 t}\right)\Big] \\
& +(\boldsymbol{\alpha}\mathbf{p}+\beta m_2)\Big[cV_2\left(\mathfrak{S}_2^{+}\mathrm{e}^{-\mathrm{i}\mathcal{E}_2 t}-\mathfrak{S}_2^{-}\mathrm{e}^{+\mathrm{i}\mathcal{E}_2 t}\right) \\
& -sV\left(\mathfrak{S}_{21}^{+}\mathrm{e}^{-\mathrm{i}\mathcal{E}_2 t}-\mathfrak{S}_{21}^{-}\mathrm{e}^{+\mathrm{i}\mathcal{E}_2 t}\right)\Big]\Big\}(-\mathrm{i}\beta)\xi(\mathbf{p})\,,
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\mathfrak{S}_{12}^{\pm} &= \int_0^t \mathrm{e}^{\pm\mathrm{i}\mathcal{E}_1 t}S_2(-\mathbf{p},t)\mathrm{d}t\,, \\
\mathfrak{S}_{21}^{\pm} &= \int_0^t \mathrm{e}^{\pm\mathrm{i}\mathcal{E}_2 t}S_1(-\mathbf{p},t)\mathrm{d}t\,, \\
\mathfrak{S}_{1,2}^{\pm} &= \int_0^t \mathrm{e}^{\pm\mathrm{i}\mathcal{E}_{1,2} t}S_{1,2}(-\mathbf{p},t)\mathrm{d}t\,.
\end{aligned} \tag{20}$$

When we derive (19) we suppose that $a_{1,2}^{(1)}(\mathbf{p},0)=b_{1,2}^{(1)}(\mathbf{p},0)=0$. It means that at $t=0$ the field distributions are determined by (13). One can find the explicit form of the functions given in (20) using (14):

$$\begin{aligned}
\mathfrak{S}_{12}^{\pm} = {} & \frac{1}{2}\Big\{\mathrm{e}^{\pm\mathrm{i}\sigma t}\frac{\sin\sigma t}{\sigma}+\mathrm{e}^{\pm\mathrm{i}\Delta t}\frac{\sin\Delta t}{\Delta} \\
& -\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_2}{\mathcal{E}_2}\left(\pm\mathrm{e}^{\pm\mathrm{i}\sigma t}\frac{\sin\sigma t}{\sigma}\mp\mathrm{e}^{\pm\mathrm{i}\Delta t}\frac{\sin\Delta t}{\Delta}\right)\Big\}(\mathrm{i}\beta)\,, \\
\mathfrak{S}_{21}^{\pm} = {} & \frac{1}{2}\Big\{\mathrm{e}^{\pm\mathrm{i}\sigma t}\frac{\sin\sigma t}{\sigma}+\mathrm{e}^{\mp\mathrm{i}\Delta t}\frac{\sin\Delta t}{\Delta} \\
& -\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_1}{\mathcal{E}_1}\left(\pm\mathrm{e}^{\pm\mathrm{i}\sigma t}\frac{\sin\sigma t}{\sigma}\mp\mathrm{e}^{\mp\mathrm{i}\Delta t}\frac{\sin\Delta t}{\Delta}\right)\Big\}(\mathrm{i}\beta)\,, \\
\mathfrak{S}_{1,2}^{\pm} = {} & \frac{1}{2}\Big\{\left(\pm\frac{1}{2\mathrm{i}\mathcal{E}_{1,2}}\mathrm{e}^{\pm 2\mathrm{i}\mathcal{E}_{1,2}t}+t\right) \\
& +\mathrm{i}\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_{1,2}}{\mathcal{E}_{1,2}}\left(\frac{1}{2\mathrm{i}\mathcal{E}_{1,2}}\mathrm{e}^{\pm 2\mathrm{i}\mathcal{E}_{1,2}t}\mp t\right)\Big\}(\mathrm{i}\beta)\,,
\end{aligned} \tag{21}$$

where

$$\sigma=\frac{\mathcal{E}_1+\mathcal{E}_2}{2}\,,\quad \Delta=\frac{\mathcal{E}_1-\mathcal{E}_2}{2}\,.$$

On the basis of (19) and (21) we obtain the expressions for the first order corrections to the vacuum case in the following form:

$$\begin{aligned}
\psi_1^{(1)}(\mathbf{r},t) = {} & \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{pr}}}{2\mathcal{E}_1}\Big\{\mathcal{E}_1\Big[sV_1 \\
& \times\left\{\frac{\sin\mathcal{E}_1 t}{\mathcal{E}_1}+t\cos\mathcal{E}_1 t-\mathrm{i}\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_1}{\mathcal{E}_1}t\sin\mathcal{E}_1 t\right\} \\
& -cV\Big\{\frac{\sin\sigma t}{\sigma}\cos\Delta t+\frac{\sin\Delta t}{\Delta}\cos\sigma t \\
& +\mathrm{i}\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_2}{\mathcal{E}_2}\sin\sigma t\sin\Delta t\left(\frac{1}{\sigma}-\frac{1}{\Delta}\right)\Big\}\Big] \\
& +(\boldsymbol{\alpha}\mathbf{p}+\beta m_1)\Big[sV_1\Big\{-\mathrm{i}t\sin\mathcal{E}_1 t \\
& +\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_1}{\mathcal{E}_1}\left(t\cos\mathcal{E}_1 t-\frac{\sin\mathcal{E}_1 t}{\mathcal{E}_1}\right)\Big\} \\
& -cV\Big\{-\mathrm{i}\sin\sigma t\sin\Delta t\left(\frac{1}{\sigma}+\frac{1}{\Delta}\right) \\
& +\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_2}{\mathcal{E}_2}\Big(\frac{\sin\Delta t}{\Delta}\cos\sigma t \\
& -\frac{\sin\sigma t}{\sigma}\cos\Delta t\Big)\Big\}\Big]\Big\}\xi(\mathbf{p})\,, \\
\psi_2^{(1)}(\mathbf{r},t) = {} & -\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{pr}}}{2\mathcal{E}_2}\Big\{\mathcal{E}_2\Big[cV_2 \\
& \times\left\{\frac{\sin\mathcal{E}_2 t}{\mathcal{E}_2}+t\cos\mathcal{E}_2 t-\mathrm{i}\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_2}{\mathcal{E}_2}t\sin\mathcal{E}_2 t\right\} \\
& -sV\Big\{\frac{\sin\sigma t}{\sigma}\cos\Delta t+\frac{\sin\Delta t}{\Delta}\cos\sigma t \\
& -\mathrm{i}\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_1}{\mathcal{E}_1}\sin\sigma t\sin\Delta t\left(\frac{1}{\sigma}+\frac{1}{\Delta}\right)\Big\}\Big] \\
& +(\boldsymbol{\alpha}\mathbf{p}+\beta m_2)\Big[cV_2\Big\{-\mathrm{i}t\sin\mathcal{E}_2 t \\
& +\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_2}{\mathcal{E}_2}\left(t\cos\mathcal{E}_2 t-\frac{\sin\mathcal{E}_2 t}{\mathcal{E}_2}\right)\Big\} \\
& -sV\Big\{\mathrm{i}\sin\sigma t\sin\Delta t\left(\frac{1}{\sigma}-\frac{1}{\Delta}\right) \\
& +\frac{\boldsymbol{\alpha}\mathbf{p}+\beta m_1}{\mathcal{E}_1}\Big(\frac{\sin\Delta t}{\Delta}\cos\sigma t \\
& -\frac{\sin\sigma t}{\sigma}\cos\Delta t\Big)\Big\}\Big]\Big\}\xi(\mathbf{p})\,.
\end{aligned} \tag{22}$$

Note that these expressions are valid for the arbitrary initial conditions $\xi(\mathbf{p})$ and exactly take into account the Lorentz invariance. It should also be mentioned that along with the harmonic functions there are several terms in the integrands which linearly depend on time. Therefore for (22) to be meaningful one has to assume that the potentials $V_{1,2}$ and V are rather weak, i.e. we study the interaction of our system with weak external fields (see (27) below).

The integrations in (22) are rather complicated for the arbitrary initial conditions. That is why we choose the function $\xi(\mathbf{p})$ analogously to Sect. 3, i.e. we again suppose that $\xi(\mathbf{p}) = (2\pi)^3\delta^3(\boldsymbol{\omega}-\mathbf{p})\xi_0$. The integrations over the momenta are eliminated with the help of the δ-functions. We also consider the high frequency approximation, i.e. $\omega \gg m_{1,2}$. Finally we obtain the following expressions for the fields distributions of the fermions ψ_k:

$$\begin{aligned}
\psi_1^{(1)}(\mathbf{r},t) = {} & \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}\frac{\mathrm{i}}{2}\Bigg\{ st[V_1 + (\boldsymbol{\alpha}\mathbf{n})V_1(\boldsymbol{\alpha}\mathbf{n})](\cos[\mathcal{E}_1(\omega)t] \\
& - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\mathcal{E}_1(\omega)t]) \\
& - c\frac{\sin[\Delta(\omega)t]}{\Delta(\omega)}[V + (\boldsymbol{\alpha}\mathbf{n})V(\boldsymbol{\alpha}\mathbf{n})](\cos[\sigma(\omega)t] \\
& - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\sigma(\omega)t])\Bigg\}\xi_0, \\
\psi_1^{(2)}(\mathbf{r},t) = {} & -\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}\frac{\mathrm{i}}{2}\Bigg\{ ct[V_2 + (\boldsymbol{\alpha}\mathbf{n})V_2(\boldsymbol{\alpha}\mathbf{n})](\cos[\mathcal{E}_2(\omega)t] \\
& - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\mathcal{E}_2(\omega)t]) \\
& - s\frac{\sin[\Delta(\omega)t]}{\Delta(\omega)}[V + (\boldsymbol{\alpha}\mathbf{n})V(\boldsymbol{\alpha}\mathbf{n})](\cos[\sigma(\omega)t] \\
& - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\sigma(\omega)t])\Bigg\}\xi_0 \, .
\end{aligned} \tag{23}$$

On the basis of (6) and (23) we can derive the first order correction to the field distribution of the fermion ν_1 in the form

$$\begin{aligned}
\nu_1^{(1)}(\mathbf{r},t) = {} & -\sin 2\theta \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}\frac{\mathrm{i}}{4}\Bigg\{ \cos 2\theta\frac{\sin\Delta(\omega)t}{\Delta(\omega)}(F_1 - F_2) \\
& \times (\cos[\sigma(\omega)t] - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\sigma(\omega)t]) \\
& - t\Bigg[F_1\Bigg\{c^2(\cos[\mathcal{E}_1(\omega)t] - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\mathcal{E}_1(\omega)t]) \\
& - s^2(\cos[\mathcal{E}_2(\omega)t] - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\mathcal{E}_2(\omega)t])\Bigg\} \\
& - F_2\Bigg\{c^2(\cos[\mathcal{E}_2(\omega)t] - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\mathcal{E}_2(\omega)t]) \\
& - s^2(\cos[\mathcal{E}_1(\omega)t] - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\sin[\mathcal{E}_1(\omega)t])\Bigg\}\Bigg]\Bigg\}\xi_0 \, ,
\end{aligned} \tag{24}$$

where $F_{1,2} = [f_{1,2}^0 - (\mathbf{f}_{1,2}\mathbf{n})(\boldsymbol{\Sigma}\mathbf{n})](1+\gamma^5)$ and $\boldsymbol{\Sigma} = -\gamma^5\boldsymbol{\alpha}$.

To calculate the intensity of the field ν_1 one should take into account that the final expression for the intensity must contain only terms linear in external fields. Therefore the first order correction to the intensity should be calculated with help of the formula

$$I_1^{(1)}(t) = \nu_1^{(0)\dagger}\nu_1^{(1)} + \nu_1^{(1)\dagger}\nu_1^{(0)} \, .$$

Using (17) and (24) we get the expression for $I_1^{(1)}$,

$$\begin{aligned}
I_1^{(1)}(t) = {} & \sin^2(2\theta)\cos 2\theta \sin[\Delta(\omega)t]\frac{1}{2} \\
& \times \left(\frac{\sin[\Delta(\omega)t]}{\Delta(\omega)} - t\cos[\Delta(\omega)t]\right) \\
& \times \big\langle\big(\big[f_2^0(\boldsymbol{\alpha}\mathbf{n}) - (\mathbf{f}_2\mathbf{n})\big] \\
& - \big[f_1^0(\boldsymbol{\alpha}\mathbf{n}) - (\mathbf{f}_1\mathbf{n})\big]\big)\left(1+\gamma^5\right)\big\rangle \, .
\end{aligned} \tag{25}$$

In (25) we use the notation $\langle(\ldots)\rangle = \xi_0^\dagger(\ldots)\xi_0$. To compute the mean value with help of the normalization spinor ξ_0 we can suppose that $\xi(\mathbf{r}) = \exp(-\mathrm{i}E_{\nu_2}t)\xi(\mathbf{r})|_{t\to 0}$. Then we notice that for spinors corresponding to high energies one has the obvious identities $(1+\gamma^5)\xi_0 \approx 2\xi_0$ and $\xi_0^\dagger(\boldsymbol{\alpha}\mathbf{n})\xi_0 \approx 1$. Putting together (18) and (25) we obtain the final expression for the intensity of the fermion ν_1:

$$\begin{aligned}
I_1(t) = {} & I_1^{(0)}(t) + I_1^{(1)}(t) \\
= {} & \sin^2(2\theta)\Bigg\{ \sin^2[\Delta(\omega)t] + \cos 2\theta\sin[\Delta(\omega)t] \\
& \times \left(\frac{\sin[\Delta(\omega)t]}{\Delta(\omega)} - t\cos[\Delta(\omega)t]\right) \\
& \times \left(\left[f_2^0 - (\mathbf{f}_2\mathbf{n})\right] - \left[f_1^0 - (\mathbf{f}_1\mathbf{n})\right]\right)\Bigg\} \, .
\end{aligned} \tag{26}$$

Using (26) it is possible to define the scope of the applied method, i.e. we can evaluate the strength of external fields necessary for the perturbative approach to be valid. With help of (26) one obtains the inequalities,

$$A\cos 2\theta \ll \Delta(\omega) \, , \quad At\cos 2\theta \ll 1 \, , \tag{27}$$

where $A = [f_2^0 - (\mathbf{f}_2\mathbf{n})] - [f_1^0 - (\mathbf{f}_1\mathbf{n})]$. If (27) is satisfied, the contribution of external axial-vector fields to neutrino flavor oscillations is small compared to the vacuum term. It should be noted that (27) is valid in the ultrarelativistic case. For neutrinos with $\mathcal{E}_k(\omega) \sim m_k$ we should rely on (22) rather than on (23). In this case the condition of the applicability of our method will be different from (27).

Now let us compare (26) with the neutrino transition probability formula. Flavor neutrinos are considered to interact with external axial-vector fields as it is described in (2)–(4). Then the probability to find muon or τ-neutrinos in the electron neutrinos beam in presence of moving and polarized matter is expressed in the following way (see, e.g., [15, 16])

$$P_{\nu_e\to\nu_{\mu,\tau}}(t) = \sin^2(2\theta_{\mathrm{eff}})\sin^2\left(\frac{\pi t}{L_{\mathrm{eff}}}\right) \, , \tag{28}$$

where

$$\sin^2(2\theta_{\mathrm{eff}}) = \frac{\Delta^2(\omega)\sin^2(2\theta)}{[\Delta(\omega)\cos 2\theta - A/2]^2 + \Delta^2(\omega)\sin^2(2\theta)} \tag{29}$$

is the definition of the effective mixing angle, and

$$\frac{\pi}{L_{\mathrm{eff}}} = \sqrt{[\Delta(\omega)\cos 2\theta - A/2]^2 + \Delta^2(\omega)\sin^2(2\theta)} \tag{30}$$

is the definition of the effective oscillation length.

To discuss the weak external field limit in (28)–(30) we should expand (29) and (30) over the small parameter A (see (27)). As a result one gets

$$\sin^2(2\theta_{\text{eff}}) \approx \sin^2(2\theta)\left(1+\frac{A\cos 2\theta}{\Delta(\omega)}\right),$$
$$\pi/L_{\text{eff}} \approx \Delta(\omega) - (A\cos 2\theta)/2\,. \tag{31}$$

It is also necessary to expand the time dependent factor in (28),

$$\sin^2\left(\frac{\pi t}{L_{\text{eff}}}\right) \approx \sin^2[2\Delta(\omega)t] - At\cos 2\theta \sin[\Delta(\omega)t]\cos[\Delta(\omega)t]\,. \tag{32}$$

Note that (32) is valid while $At\cos 2\theta \ll 1$, which coincides with the second inequality in (27). With the help of (31) and (32) the neutrino transition probability is reduced to the form

$$P_{\nu_e\to\nu_{\mu,\tau}}(t) = \sin^2(2\theta)\Bigg\{\sin^2[\Delta(\omega)t] + \cos 2\theta \sin[\Delta(\omega)t] \times\left(\frac{\sin[\Delta(\omega)t]}{\Delta(\omega)} - t\cos[\Delta(\omega)t]\right) \times\left(\left[f_2^0-(\mathbf{f}_2\mathbf{n})\right]-\left[f_1^0-(\mathbf{f}_1\mathbf{n})\right]\right)\Bigg\},$$

which coincides with (26). This comparison shows that neutrino flavor oscillations in weak axial-vector fields (e.g., if a neutrino propagates in moving and polarized matter) can be treated with help of the classical field theory approach.

5 Conclusion

In conclusion we mention that the evolution of coupled classical fermions under the influence of external axial-vector fields has been studied in the present paper. We have discussed the particular case of two coupled fermions and formulated the Cauchy problem for this system. If the initial conditions were chosen in the appropriate way, as it has been shown in Sect. 2, the described system might serve as a theoretical model of neutrino flavor oscillations in matter. The initial condition problem has been solved with help of perturbation theory. We have found the zero and the first order terms in the field distributions expansions over the external fields strength. It should be noted that the obtained results exactly take into account Lorentz invariance, and also they are valid in $(3+1)$-dimensional space-time. The intensity of the zero order term corresponds to the case of the neutrino flavor oscillations in vacuum. Therefore we have generalized our previous calculations performed in $(1+1)$ dimensions in [9]. The first order correction is responsible for the neutrino interaction with moving and polarized matter. We have obtained this intensity of the fermion field at great oscillation frequencies of the initial field distribution, that corresponds to ultrarelativistic neutrinos. Note that we have compared our results with the transition probability formula for neutrino flavor oscillations in moving and polarized matter as derived in [15, 16] and revealed agreement in the case of weak external fields. This comparison proves the validity of the method elaborated in our work. Finally it has been demonstrated that neutrino flavor oscillations in moving and polarized matter could be described with the help of the classical field theory.

It is interesting to notice that along with the usual neutrino oscillations phase equal to $\Delta m^2/(4\omega)$ the classical field theory approach also reveals rapid harmonic oscillations on the frequency (see, e.g., (15) and (22))

$$\omega_{\text{rapid}} = \frac{\mathcal{E}_1(\omega)+\mathcal{E}_2(\omega)}{2} \to \omega + \frac{m_1^2+m_2^2}{4\omega}\,.$$

However these terms are suppressed by the ratios m_k/ω which are small for large values of ω. This case corresponds to ultrarelativistic neutrinos. The analogous terms were discussed in many previous publications devoted to neutrino flavor oscillations (see, e.g., [6, 8]). In our work it has been demonstrated that such contributions to the neutrino transition probability appear even in the classical field theory approach. These terms arise from an accurate account of Lorentz invariance.

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