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# Evolution of coupled classical fields

Maxim Dvornikov

*Fachbereich Physik, Universität Rostock, 18051 Rostock, Germany*

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**Abstract**

We study the evolution of the coupled scalar and fermion fields within the classical field theory. We examine the case of $N$ coupled fields in $(1+3)$-dimensional space. The general expressions for the fields distributions are obtained. The particular case of two fields in $(1+1)$-dimensional space is carefully studied. We obtain the expressions for the averaged fields intensities and show that in the relativistic limit they are similar to the usual transition probabilities formulae of neutrino oscillations.


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The particle mixing plays an important role in elementary particle physics. According to the experimental data the particle mixing exists in both quark and lepton sectors of the standard model. The idea of mixing among the two quark flavors was put forward in Ref. [1] to explain the baryons decays. The mixing in the leptonic sector of the standard model was proposed in Ref. [2]. In that parer the neutrino mixing and oscillations were studied on the analogy of the known at that time $K^0 \leftrightarrow \bar{K}^0$ oscillations. Then in Ref. [3] this approach was generalized on the mixing between three neutrino flavors. Recently we obtained the strong evidence in favor of neutrino oscillations and, therefore, mixing (see, for instance, Ref. [4]). For example, neutrino oscillations are likely to be the most plausible explanation of the solar and atmospheric neutrino problems.

In Ref. [2] the neutrino oscillations were examined within the quantum mechanical approach. Schrödinger like differential equation for the description of the two-level neutrino system was proposed. On the basis of this equation one can derive the famous transition probability formula,

$$P(t) = \sin^2(2\theta_{\text{vac}}) \sin\left(\frac{\Delta m^2}{4E} t\right), \tag{1}$$

*E-mail address:* maxim.dvornikov@uni-rostock.de (M. Dvornikov).

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where $\theta_{\mathrm{vac}}$ is the vacuum mixing angle, $\Delta m^2$ is the mass squared difference and $E$ is the energy of the system. Up to now Eq. (1) is of use in numerous phenomenological studies of neutrino oscillations. However more profound analysis of particle mixing and oscillations is necessary. The approach to the oscillations phenomenon based on the field theory methods should be elaborated.

In the last decade a great deal of studies on the field theoretical substantiation of Eq. (1) were carried out. First of all it is necessary to mention works by M. Blasone and G. Vitiello and their collaborators (see Refs. [5–8]). In these papers the authors made the comprehensive analysis of the fermion and boson mixing transformations using the methods of quantum field theory. It was demonstrated that the vacuum structure of mass eigenstates is not equivalent to one of flavor eigenstates. The quantum mechanical formula for the transition probability was also reproduced. Moreover some corrections, which result from more careful quantum field theory analysis, were obtained. In Ref. [9] the group theoretical aspects of neutrino oscillations were discussed. Analogous approach to description of the neutrino oscillations was developed in Refs. [10,11].

Rather appreciable contributions to the investigation of the flavor neutrino oscillations were made in Refs. [12,13]. In those papers the wave packages treatment of neutrino oscillations was developed as well as the discussion of the neutrino oscillations phase is presented. The neutrino oscillations phase was also studied in Ref. [14]. A very interesting approach to the description of the neutrino flavor oscillations was proposed in Ref. [15]. In that paper the covariant path amplitudes method was applied for the analysis of the neutrino oscillations phase.

Recently we elaborated the quasi-classical approach for the description of spin (see Refs. [16–18]) and flavor neutrino oscillations (see Ref. [19]). It was shown that neutrino oscillations in moving and polarized matter under the influence of arbitrary electromagnetic fields were described by the generalized Lorentz invariant quasi-classical Bargmann–Michel–Telegdi equation. It is interesting to note that the equation describing the precession of the neutrino three-dimensional spin vector (neutrino spin in particle's rest frame) is the usual Bloch equation. We demonstrated that neutrino spin rotates around a certain direction determined by the velocities and polarizations of background fermions as well as the electromagnetic field strength.

The method involving the Bloch equation for the treatment of neutrino flavor oscillations was proposed in Ref. [20]. If one considers the evolution of two neutrinos system (e.g., $\nu_e$ and $\nu_\mu$), it is possible to introduce the "polarization" vector $\mathbf{P} = \mathrm{Tr}(\boldsymbol{\sigma}\rho)$, where $\boldsymbol{\sigma}$ are the Pauli matrices and $\rho$ is the $2\times 2$ density matrix. If neutrinos propagate in vacuum, the vector $\mathbf{P}$ was shown to precess without loss of length according to the Bloch equation. The appearance of classical effects in various quantum systems (including the analysis of a two-level system with help of the Bloch equation) was discussed in Ref. [21]. Thus basing on the similarity in the description of neutrino spin and flavor oscillations we suppose that classical theory methods could have been applied for the treatment of the flavor oscillations. However this supposition should be substantiated by the direct calculations that show the classical theory yields at least the same results as the quantum one.

In this Letter we study the evolution of the coupled scalar as well as fermion fields within the context of classical field theory. The main goal of our article is to demonstrate that neutrino oscillations can be described within the classical approach. The classical approach was also adopted since we should not be puzzled by a problem: must we rely on flavor or mass eigenstates in our treatment of neutrino oscillations? The intensive discussion about this topic takes place in Refs. [13,22]. The case of $N$ coupled fields in $(1+3)$-dimensional space is examined. We solve the Cauchy problem for this system, i.e., for the given initial conditions we find the fields distributions for any time point. In order to analyze the obtained expressions we study the particular case of two fields in $(1+1)$-dimensional space. For the specific initial conditions the expressions for the averaged fields intensities are obtained. We also show that in the relativistic limit they are similar to the usual transition probabilities formulae of neutrino oscillations in vacuum. It is interesting to mention that the expressions for the averaged fields intensities are identical for both bosons and fermions.

First let us discuss the case of $N$ arbitrary coupled scalar fields. For simplicity we suppose that these fields are the real ones. The Lagrangian for this system

is expressed in the following form

$$\mathcal{L}(\boldsymbol{\varphi}) = \sum_{k=1}^{N} \mathcal{L}_0(\varphi_k) + \sum_{\substack{i,j=1 \\ i\neq k}}^{N} g_{ik}\varphi_i\varphi_k, \tag{2}$$

where $g_{ik}$ are the coupling constants, $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$, and

$$\mathcal{L}_0(\varphi_k) = \frac{1}{2}\partial_\mu\varphi_k\partial^\mu\varphi_k - \frac{\mathfrak{m}_k^2}{2}\varphi_k^2, \tag{3}$$

is the Lagrangian for the field $\varphi_k(\mathbf{r}, t)$ at the absence of the additional coupling, $\mathfrak{m}_k$ is the mass corresponding to this field. It is necessary to note that the second term in Eq. (2) is assumed to be an interaction between fields $\varphi_k$.

In order to describe the evolution of the system (2), (3) we should set the Cauchy problem for this system. For the initial conditions,

$$\varphi_i(\mathbf{r}, 0) = f_i(\mathbf{r}), \qquad \dot{\varphi}_i(\mathbf{r}, 0) = g_i(\mathbf{r}), \tag{4}$$

where $f_i(\mathbf{r})$ and $g_i(\mathbf{r})$ are the given functions, one should find the fields distributions $\varphi_k(\mathbf{r}, t)$ for any time point.

It is always possible to diagonalize the Lagrangian (2) with help of the transformation,

$$\varphi_i(\mathbf{r}, t) = \sum_{k=1}^{N} M_{ik}u_k(\mathbf{r}, t).$$

Thus the Lagrangian expressed in terms of the fields $u_k(\mathbf{r}, t)$ takes the form

$$\mathcal{L}(\mathbf{u}) = \sum_{k=1}^{N} \mathcal{L}_0(u_k),$$

where $\mathcal{L}_0(u_k)$ is the Lagrangian for the field $u_k(\mathbf{r}, t)$,

$$\mathcal{L}_0(u_k) = \frac{1}{2}\partial_\mu u_k\partial^\mu u_k - \frac{m_k^2}{2}u_k^2,$$

and $m_k$ are the corresponding masses. It should be noted that these masses differ from the masses of the fields $\varphi_k$. The fields $u_k(\mathbf{r}, t)$ are usually called mass eigenstates in contrast to $\varphi_k(\mathbf{r}, t)$.

One can write the differential equations for the fields $u_k(\mathbf{r}, t)$. It is the system of the usual homogeneous Klein–Gordon equations. Their solutions have the form

$$u_k(\mathbf{r}, t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\big[a_k^+(\mathbf{p})e^{-i\mathcal{E}_k t} + a_k^-(\mathbf{p})e^{i\mathcal{E}_k t}\big]e^{i\mathbf{pr}}, \tag{5}$$

where $\mathcal{E}_k = \sqrt{\mathbf{p}^2 + m_k^2}$, and $a_k^\pm(\mathbf{p})$ are the Fourier coefficients. Note that $a_k^\pm(\mathbf{p})$ are the $c$-numbers.

To solve the Cauchy problem we introduce the functions

$$F_k(\mathbf{r}) = \sum_{i=1}^{N}\big(M^{-1}\big)_{ki} f_i(\mathbf{r}),$$

$$G_k(\mathbf{r}) = \sum_{i=1}^{N}\big(M^{-1}\big)_{ki} g_i(\mathbf{r}).$$

These functions are the initial conditions for the fields $u_k$. Then one should pick out the coefficients $a_k^\pm(\mathbf{p})$ so that to satisfy the initial conditions (4). From Eqs. (4) and (5) we obtain

$$a_k^\pm(\mathbf{p}) = \frac{1}{2}\left(F_k(\mathbf{p}) \pm i\frac{G_k(\mathbf{p})}{\mathcal{E}_k}\right),$$

where $F_k(\mathbf{p})$ and $G_k(\mathbf{p})$ are the Fourier transforms of the functions $F_k(\mathbf{r})$ and $G_k(\mathbf{r})$, respectively. Finally we receive the fields distributions $\varphi_j(\mathbf{r}, t)$ in the explicit form

$$\varphi_j(\mathbf{r}, t) = \sum_{ik=1}^{N} M_{jk}\big(M^{-1}\big)_{ki} \times \int \mathrm{d}^3\mathbf{r}'\big[\dot{D}_k(\mathbf{r}-\mathbf{r}', t)f_i(\mathbf{r}') + D_k(\mathbf{r}-\mathbf{r}', t)g_i(\mathbf{r}')\big], \tag{6}$$

where

$$D_k(\mathbf{r}, t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{pr}}\frac{\sin\mathcal{E}_k t}{\mathcal{E}_k}, \tag{7}$$

is the Pauli–Jordan function. It is interesting to list some of the properties of the Pauli–Jordan function

$$D_k(\mathbf{r}, 0) = 0, \qquad \dot{D}_k(\mathbf{r}, 0) = \delta^3(\mathbf{r}),$$
$$\ddot{D}_k(\mathbf{r}, 0) = 0.$$

It is worth noticing that the initial conditions in Eq. (4) are consistent with these properties of the Pauli–Jordan function. We also mention that the Pauli–Jordan function can be expressed in the explicit form

(see, e.g., Ref. [23]),

$$D_k(\mathbf{r},t)=\frac{1}{2\pi}\varepsilon(t)\delta\bigl(s^2\bigr)-\frac{m_k}{4\pi s}\varepsilon(t)\theta\bigl(s^2\bigr)J_1(m_k s), \tag{8}$$

where $s^2=t^2-\mathbf{r}^2$ and

$$\varepsilon(t)=\begin{cases}1, & t>0,\\ -1, & t<0,\end{cases}\qquad \theta(s)=\begin{cases}1, & s>0,\\ 0, & s<0\end{cases}$$

are the step functions. Thus Eqs. (6) and (7) represent the exact solution of the Cauchy problem for arbitrary functions $f_i(\mathbf{r})$ and $g_i(\mathbf{r})$.

The integrals calculation in Eq. (6), however, are rather awkward in general $(1+3)$-dimensional space. Thus let us, for simplicity, consider the space with $1+1$ dimensions. Instead of Eq. (7) we have

$$D_k(x,t)=\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}e^{ipx}\frac{\sin\mathcal{E}_k t}{\mathcal{E}_k}.$$

Now $\mathcal{E}_k=\sqrt{p^2+m_k^2}$. One can also obtain the Pauli–Jordan function in the explicit form (see Eq. (8)) in $(1+1)$-dimensional space,

$$D_k(x,t)=\frac{1}{2}\theta\bigl(s^2\bigr)J_0(m_k s), \tag{9}$$

$$\dot{D}_k(x,t)=t\delta\bigl(s^2\bigr)-\frac{m_k t}{2s}\theta\bigl(s^2\bigr)J_1(m_k s). \tag{10}$$

Here $s^2=t^2-x^2$.

We suppose that $g_k(x)=0$ and $f_k(x)\neq 0$. Then we receive (see also Ref. [24])

$$\begin{aligned}\varphi_j(x,t)=\sum_{ik=1}^{N}M_{jk}\bigl(M^{-1}\bigr)_{ki}&\\ \times\left\{\frac{1}{2}\bigl[f_i(x-t)+f_i(x+t)\bigr]\right.&\\ \left.-\frac{m_k t}{2}\int\limits_{x-t}^{x+t}\mathrm{d}y\ f_i(y)\frac{J_1(m_k s)}{s}\right\}.&\end{aligned} \tag{11}$$

It should be noted that if functions $f_i(x)\neq 0$ in a bounded region and $f_i(x)\to 0$, when $x\to\pm\infty$ then the second term in Eq. (11) is the vanishing one and

$$\varphi_j(x,t)\to\frac{1}{2}\bigl[f_j(x-t)+f_j(x+t)\bigr].$$

This property of the Klein–Gordon equation was also mentioned in Ref. [24]. The described feature has one interesting physical implication. If a single particle appears far from a detector, then its field distribution is localized in space. When a particle begins propagating towards a detector its field distribution approaches to the initial conditions. Thus the effect of various nontrivial phenomena (like conversion, or oscillations, from one field type to another) will be vanishing.

Now let us choose the initial conditions. We suppose $f_1(x)=0$ and

$$f_2(x)=\mathfrak{A}\sin\left(\frac{\omega}{2}x\right),\quad \mathfrak{A}=\frac{4}{\sqrt{\omega L}},$$

where $L$ is the "volume" of the space. Note that $\mathfrak{A}$ is just the normalization factor. In this case we can calculate the integral in Eq. (11) explicitly

$$\begin{aligned}&\int\limits_{x-t}^{x+t}\mathrm{d}y\sin\left(\frac{\omega}{2}y\right)\frac{J_1(ms)}{s}\\ &\quad=\pi\sin\left(\frac{\omega}{2}x\right)J_{1/2}\left(\frac{t}{2}\beta_1\right)J_{1/2}\left(\frac{t}{2}\beta_2\right),\end{aligned} \tag{12}$$

where

$$\beta_{1,2}=\sqrt{\frac{\omega^2}{4}+m^2}\mp\frac{\omega}{2}.$$

In computation of the integral in Eq. (12) we used the expression

$$\begin{aligned}&\int\limits_0^a\mathrm{d}x\frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}}J_\nu(cx)\\ &\quad=\frac{\pi}{2}J_{\nu/2}\left[\frac{a}{2}\bigl(\sqrt{b^2+c^2}-b\bigr)\right]\\ &\qquad\times J_{\nu/2}\left[\frac{a}{2}\bigl(\sqrt{b^2+c^2}+b\bigr)\right].\end{aligned}$$

One half-order Bessel function can be expressed in terms of the elementary function. Namely,

$$J_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sin z. \tag{13}$$

Let us consider the case when $\omega$ has great values compared to the masses $m_{1,2}$: $\omega\gg m_{1,2}$. This situation corresponds to the high energy approximation or relativistic "particles". Then, the parameters $\beta_{1,2}$ take

the form

$$\beta_1 = \frac{m^2}{\omega}, \qquad \beta_2 = \omega. \tag{14}$$

The field distribution $\varphi_1(x,0)$ is equal to zero. Thus we can describe the dynamics of this field for the subsequent points of time. If one studies the evolution of two fields, $\varphi_{1,2}(x,t)$, it is possible to parameterize the matrix $M_{jk}$ with help of one angle

$$M_{jk} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{15}$$

Using Eqs. (12)–(15) we can rewrite Eq. (11) in the following way

$$\begin{aligned}\varphi_1(x,t) = {} & 2\mathfrak{A}\sin 2\theta \sin\left(\frac{\omega}{2}x\right)\sin\left(\frac{\omega}{2}t\right) \\ & \times \sin\left[\frac{t}{4\omega}\left(m_1^2 - m_2^2\right)\right] \\ & \times \cos\left[\frac{t}{4\omega}\left(m_1^2 + m_2^2\right)\right].\end{aligned} \tag{16}$$

Now let us discuss the field measurement process. In case of rapidly varying fields ($\omega \gg m_{1,2}$), a detector registers not the field strength, but the intensity of the field which is proportional to the field strength squared, $I \sim \varphi^2(x,t)$. Moreover a detector has limited sensitivity, i.e., it cannot register arbitrary field variation in time and in space. Thus we should average the intensity over the characteristic time and space scales. These scales should be greater than typical time and space scales of the field in question, i.e., $1/\omega$.

To calculate the mean value of the intensity one should take into account the expressions

$$\left\langle \sin^2\left(\frac{\omega}{2}x\right)\right\rangle = \left\langle \sin^2\left(\frac{\omega}{2}t\right)\right\rangle = \frac{1}{2},$$

and

$$\begin{aligned}\left\langle \sin^2\left[\frac{t}{4\omega}\left(m_1^2 - m_2^2\right)\right]\right\rangle &= \sin^2\left[\frac{t}{4\omega}\left(m_1^2 - m_2^2\right)\right], \\ \left\langle \cos^2\left[\frac{t}{4\omega}\left(m_1^2 + m_2^2\right)\right]\right\rangle &= \cos^2\left[\frac{t}{4\omega}\left(m_1^2 + m_2^2\right)\right],\end{aligned}$$

since $(m_1^2 \pm m_2^2)/\omega \ll \omega$. Then we should introduce the normalized intensity of the field $\varphi_1(x,t)$ according to the formula

$$P(t) = \frac{\langle I\rangle(t)}{\mathfrak{A}^2}.$$

Finally we obtain the expression for $P(t)$ in the following form,

$$\begin{aligned}P(t) = {} & \sin^2 2\theta \sin^2\left(\frac{\Delta m^2}{4\omega}t\right) \\ & \times \left\{1 - \sin^2\left(\frac{m_1^2 + m_2^2}{4\omega}t\right)\right\},\end{aligned} \tag{17}$$

where we introduced the common notation $\Delta m^2 = m_1^2 - m_2^2$.

It is necessary to identify the $\omega$ parameter. We cannot directly equate it to the particle energy, $E = \hbar\omega$, since we are using the classical approach here. Moreover, the chosen "wave function", $f_2(x) \sim \sin(\omega x/2)$, does not correspond to a definite momentum and thus to a definite energy. However we can calculate the averaged energy density of the system,

$$\langle \rho_E\rangle = \left\langle \frac{1}{2}\left\{\left(\frac{\mathrm{d}f_2}{\mathrm{d}x}\right)^2 + m_2^2 f_2^2(x)\right\}\right\rangle. \tag{18}$$

Here we suppose that $f_1(x) = 0$. Using Eq. (18) in relativistic limit ($\omega \gg m_{1,2}$) we obtain that

$$\langle \rho_E\rangle = \frac{\omega}{L},$$

and we can identify $\omega$ with the energy of the system. Thus the first term in Eq. (17) is similar to the well-known formula for the transition probability in the two neutrino system. It is interesting to mention that the second term in Eq. (17) contains the harmonic oscillations with the frequency $(m_1^2 + m_2^2)/4\omega$. Analogous additional term was obtained in Refs. [5–8] and was treated as the quantum field theory correction to the Eq. (1). However our approach demonstrates that this term appears when one uses classical field theory. It results from the accurate account of the Lorentz invariance.

Now let us discuss the case of the coupled fermion fields. The Lagrangian for this system is expressed in the following way

$$\mathcal{L}(\boldsymbol{\nu}) = \sum_{k=1}^{N} \mathcal{L}_0(\nu_k) + \left(\sum_{\substack{i,j=1 \\ i>k}}^{N} g_{ik}\bar{\nu}_i\nu_k + \text{h.c.}\right),$$

where $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_N)$, and

$$\mathcal{L}_0(\nu_k) = \bar{\nu}_k\left(i\gamma^\mu\partial_\mu - \mathfrak{m}_k\right)\nu_k.$$

We again should set the Cauchy problem for the system of differential equations in question. However here one has to impose only one initial condition since Dirac equation is the first-order differential equation,

$$\nu_k(\mathbf{r},0)=\xi_k(\mathbf{r}). \tag{19}$$

Analogously to the case of the scalar fields we can introduce the mass eigenstates,

$$\nu_i(\mathbf{r},t)=\sum_{k=1}^{N} M_{ik}\psi_k(\mathbf{r},t).$$

The Lagrangian expressed in terms of the mass eigenstates has the form,

$$\mathcal{L}(\boldsymbol{\psi})=\sum_{k=1}^{N}\mathcal{L}_0(\psi_k),$$

where

$$\mathcal{L}_0(\psi_k)=\bar{\psi}_k\bigl(i\gamma^\mu\partial_\mu-m_k\bigr)\psi_k,$$

is the Lagrangian for the field $\psi_k(\mathbf{r},t)$. Note that masses $m_k$ differ from the masses of the fields $\nu_k$.

The solution of the Dirac equations for the mass eigenstates fields can be expressed in the following way,

$$\psi_k(\mathbf{r},t)=\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}}\bigl[a_s(\mathbf{p})u_s(\mathbf{p})e^{-i\mathcal{E}_k t}+b_s(-\mathbf{p})v_s(-\mathbf{p})e^{i\mathcal{E}_k t}\bigr]e^{i\mathbf{pr}}. \tag{20}$$

Here $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ are the basis spinors, $a_s(\mathbf{p})$ and $b_s(\mathbf{p})$ are the indeterminate functions.

Now we should find the values of the $a_s(\mathbf{p})$ and $b_s(\mathbf{p})$ $c$-number functions to satisfy the initial condition given in Eq. (19). The calculations are analogous to the previously discussed case of the coupled scalar fields. Thus we arrive to the solution of the Dirac equations which are valid for arbitrary functions $\xi_i(\mathbf{r})$,

$$\nu_j(\mathbf{r},t)=\sum_{ik=1}^{N}M_{jk}\bigl(M^{-1}\bigr)_{ki}\times\int\mathrm{d}^3\mathbf{r}'\,S_k(\mathbf{r}'-\mathbf{r},t)\bigl(-i\gamma^0\bigr)\xi_i(\mathbf{r}'), \tag{21}$$

where

$$S_k(\mathbf{r},t)=\bigl(i\gamma^\mu\partial_\mu+m_k\bigr)D_k(\mathbf{r},t),$$
$$x^\mu=(t,\mathbf{r}),$$

is the Pauli–Jordan function for a fermion field (see, e.g., Ref. [25]). In deriving of Eq. (21) we used the orthonormality conditions

$$u_s^\dagger(\mathbf{p})u_r(\mathbf{p})=v_s^\dagger(\mathbf{p})v_r(\mathbf{p})=\delta_{sr},$$
$$u_s^\dagger(\mathbf{p})v_r(-\mathbf{p})=v_s^\dagger(\mathbf{p})u_r(-\mathbf{p})=0,$$

and the formulae for the summation over the spin indexes

$$\sum_s u_s(\mathbf{k})u_s^\dagger(\mathbf{k})=\frac{\not{p}+m}{2p^0}\gamma^0,$$
$$\sum_s v_s(\mathbf{k})v_s^\dagger(\mathbf{k})=\frac{\not{p}-m}{2p^0}\gamma^0.$$

It is interesting to mention that the function $S_k(\mathbf{r},t)$ has the following property,

$$S_k(\mathbf{r},0)=i\gamma^0\delta^3(\mathbf{r}).$$

Thus the solution given in Eq. (21) is consistent with the initial condition (19). Eq. (21) can be rewritten in the non-covariant form which is, however, more convenient for the further analysis,

$$\nu_j(\mathbf{r},t)=\sum_{ik=1}^{N}M_{jk}\bigl(M^{-1}\bigr)_{ki}\times\Bigl\{-\int\mathrm{d}^3\mathbf{r}'\,(\boldsymbol{\alpha}\nabla_{\mathbf{r}})D_k(\mathbf{r}-\mathbf{r}',t)\xi_i(\mathbf{r}')+\int\mathrm{d}^3\mathbf{r}'\,\dot{D}_k(\mathbf{r}-\mathbf{r}',t)\xi_i(\mathbf{r}')-im_k\beta\int\mathrm{d}^3\mathbf{r}'\,D_k(\mathbf{r}-\mathbf{r}',t)\xi_i(\mathbf{r}')\Bigr\}, \tag{22}$$

where we use common notations for the gamma matrixes, $\boldsymbol{\alpha}=\gamma^0\boldsymbol{\gamma}$ and $\beta=\gamma^0$.

Just for simplicity we again discuss the case of two coupled Dirac fields in the space with $1+1$ dimensions. Dirac equation in $(1+1)$-dimensional space was carefully studied in Refs. [26–28]. The gamma matrixes have the form,

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad\gamma^1=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{23}$$

Now one should set the initial conditions. Let us assume that $\xi_1(x)=0$ and $\xi_2(x)$ is expressed in the following way,

$$\xi_2(x)=\frac{1}{2}\begin{pmatrix}\cos(\omega x/2)\\\sin(\omega x/2)\end{pmatrix}. \tag{24}$$

We obtained the expression for the evolution of the $\nu_1(x,t)$ which accounts for the exact dependencies on the particles masses. However it appeared to be rather awkward. Nevertheless it can be shown that the third term in Eq. (22) is negligible in $(1+1)$-dimensional space. Indeed, let us consider, for instance, the integral

$$I(x,t) = \int\limits_{-\infty}^{+\infty} \mathrm{d}y\, D_k(x-y,t) \sin\left(\frac{\omega}{2}y\right). \tag{25}$$

We remind that the Pauli–Jordan function in $(1+1)$-dimensional space is given in Eq. (9). The integral in Eq. (25) can be calculated explicitly and expressed in the form

$$I(x,t) = \sin\left(\frac{\omega}{2}x\right) \frac{\sin(t\sqrt{m_k^2 + (\omega/2)^2})}{\sqrt{m_k^2 + (\omega/2)^2}}. \tag{26}$$

Here we used the known value of the integral

$$\int\limits_0^a \mathrm{d}x\, J_0\bigl(b\sqrt{a^2-x^2}\bigr)\cos(cx) = \frac{\sin(a\sqrt{b^2+c^2})}{\sqrt{b^2+c^2}}.$$

Thus in the high energy approximation ($\omega \gg m_k$) we obtain that $I(x,t) \to 0$. It is also interesting to note that one should carefully follow the order of integration and differentiation while using the Pauli–Jordan function. Indeed using, for example, Eqs. (10)–(12) and (26) we can see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{-\infty}^{+\infty} \mathrm{d}x\, D_k(x,t) f(x) \neq \int\limits_{-\infty}^{+\infty} \mathrm{d}x\, \frac{\partial}{\partial t} D_k(x,t) f(x),$$

because Pauli–Jordan function is the singular one.

Finally we get the expression for the $\nu_1(x,t)$,

$$\begin{aligned}\nu_1(x,t) = {} & \sin 2\theta \left[\sin\left(\frac{\omega}{2}t\right) + \cos\left(\frac{\omega}{2}t\right)\right] \\ & \times \sin\left[\frac{t}{4\omega}\bigl(m_1^2 - m_2^2\bigr)\right] \cos\left[\frac{t}{4\omega}\bigl(m_1^2 + m_2^2\bigr)\right] \\ & \times \begin{pmatrix} \cos(\omega x/2) \\ \sin(\omega x/2) \end{pmatrix}. \end{aligned} \tag{27}$$

In deriving of Eq. (27) we used the fact that

$$\frac{\partial}{\partial x} D_k(x,t) = -\frac{x}{t}\frac{\partial}{\partial t} D_k(x,t). \tag{28}$$

The measurable quantity of the classical Dirac field is the intensity. It is proportional to the $|\nu_1(x,t)|^2$. However we again should average the intensity over space and time. Thus, using Eq. (27) we obtain for $\langle I\rangle(t)$ the following expression

$$\begin{aligned}\langle I\rangle(t) = {} & \sin^2 2\theta \sin^2\left(\frac{\Delta m^2}{4\omega}t\right) \\ & \times \left\{1 - \sin^2\left(\frac{m_1^2 + m_2^2}{4\omega}t\right)\right\}, \end{aligned} \tag{29}$$

which coincides with the similar expression derived for the scalar field. Note that Eq. (29) again contains the additional term oscillating with the frequency $(m_1^2 + m_2^2)/4\omega$.

The calculations performed in this Letter demonstrate (especially Eq. (29)) that neutrino flavor oscillations can be treated in frames of the classical theory. According to the classical field theory approach the evolution of flavor neutrinos is described in the following way.

(1) Flavor neutrino emission in a reaction. This process can be described by means of the quantum approach. However here we should obtain the final field distribution rather than the emission probability. It is also possible to admit that the mixture of the neutrino flavors appears in a process, as it was proposed in Ref. [22]. In this case one should set other initial conditions in Eqs. (4) and (19);

(2) Neutrino propagation towards a detector. One can successfully use the methods elaborated in this Letter for the description of the neutrino conversion or oscillations. Basing on the initial fields distributions obtained in the item (1) we derive the final fields distributions that take into particles mixing;

(3) Neutrino interaction with a detector. This process again can be described by means of quantum field theory. On the basis of the fields distributions obtained in the item (2) one can calculate the neutrino flux measured with a detector.

Thus we should not directly involve the mass eigenstates if we are using classical approach.

In conclusion we mention that the evolution of the coupled scalar as well as fermion fields within the classical field theory has been studied in this Letter. We have examined the case of $N$ coupled fields in

$(1 + 3)$-dimensional space. The Cauchy problem has been formulated for these systems. We have solved it for arbitrary initial conditions. The particular case of two coupled fields in $(1 + 1)$-dimensional space has been studied. Finally we have obtained the expressions for the averaged fields intensities. It has been shown that in the relativistic limit these expressions were similar to the usual transition probabilities formulae of neutrino oscillations in vacuum. The discussion of the additional terms in transition probabilities formulae has been presented. It has been demonstrated that the expressions for the averaged fields intensities for both bosons and fermions turned out to be identical. We have shown by means of the direct calculations that the flavor oscillations phenomenon could be described within the classical approach. Thus one can conclude that the usage of the quantum mechanics is inexpedient because classical field theory yields more elegant description of the problem in question.

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