# Higher Order Predicted Terms for some QCD observables, Using various Scale Optimization procedures

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## Introduction

- The principle of minimum sensitivity (PMS) is reviewed. It is possible to expand the optimized quantities in terms of the quantities which exist in the standard perturbative series for the observables. It is then possible to obtain the predicted higher order terms.

- The calculations indicate that at the moment, just the NLO predicted term, is unique.

- The CORGI approach which is based on resuming the ultraviolet terms, is introduced.

- Considering the definition of coupling constant and also the expressions for RS invariants quantities in this approach, it is possible to obtain the predicted terms in higher order approximations. The predicted terms are unique.

# Principle of Minimum Sensitivity

- For PMS method in the NLO approximation, a QCD observable R can be written as:

$$R^{(2)} = a(\tau)(1 + r_1 a(\tau)) \tag{1}$$

where $\tau$ is defined by $\tau = b \log(\frac{\mu}{\tilde{\Lambda}})$.

- The following property always exist:

$$\frac{\partial R^{(i)}}{\partial (RS)}_{|RS=\texttt{Optimized RS}} = 0 \;, \tag{2}$$

where $(RS)$ involves all quantities which use to parameterize the scheme dependence of the observable.

- For $R^{(2)}$: $\frac{\partial R^{(2)}}{\partial \tau}_{|\tau=\bar{\tau}} = 0$.

Using the QCD $\beta$-function :

$$\frac{\partial a^{(2)}}{\partial \tau} = \frac{\beta^2(a)}{b} = -a^2(1 + ca) \;, \tag{3}$$

therefore:

$$\frac{\partial R^{(2)}}{\partial \tau} = -a^2(1 + ca)(1 + 2r1a) + a^2 \frac{\partial r_1}{\partial \tau} \;. \tag{4}$$

The orders $a^2$-terms must cancel for the formal self-consistency of the perturbation theory.

- As the implication

$$\frac{\partial r_1}{\partial \tau} = 1 \ . \qquad (5)$$

So $\rho_1 = \tau - r_1(\tau)$ is a constant, independent of the un-physical variable $\tau$. The PMS criterion requires that the Eq.(4) should vanish at $\tau{=}\bar{\tau}$.

Eq.(4), $\frac{\partial r_1}{\partial \tau} = 1$ & PMS Criterion $\Rightarrow$

$$\frac{\partial R^{(2)}}{\partial \tau} = -a^2(1+ca)(1+2r_1 a) + (a^2 \times 1) = 0 \qquad (6)$$

$$[2\bar{r}_1(1+c\bar{a}) + c]|_{\tau=\bar{\tau}} = 0 \quad \Rightarrow \quad \bar{r}_1 = \frac{-c}{2(1+c\bar{a})} \qquad (7)$$

where $\bar{r}_1 = r_1(\bar{\tau})$, $\bar{a} = a(\bar{\tau})$.
Finally we arrive at:

$$R_{opt} = \bar{a}(1+\bar{r_1}\bar{a}) = \bar{a}[\frac{1+\frac{1}{2}c\bar{a}}{1+c\bar{a}}] \ . \qquad (8)$$

We should find $\bar{a}$ in terms of $a$ and $r_1$. Expansion the result in terms of $a$, will be predicted the $r_2$ term.
Invariant quantity $\rho_1$:

$$\tau - r_1(\tau) = \bar{\tau} - \bar{r}_1(\bar{\tau}) \ . \qquad (9)$$

$\tau = b\log(\frac{\mu}{\tilde{\Lambda}})$ is equal to $\frac{1}{a}$ in one loop so, so:

$$\frac{1}{a} - r_1(\tau) = \frac{1}{\bar{a}} - \bar{r}_1(\bar{\tau}) \qquad (10)$$

From (7) and (10):

$$\frac{1}{a} - r_1(\tau) - \frac{1}{\bar{a}} + \frac{c}{2(1+c\bar{a})} = 0\ . \tag{11}$$

Solutions:

$$\bar{a}_{1,2} = \quad -\frac{1}{4}\left[\frac{3ac-2+2r_1a}{c(-1+r_1a)} \pm \frac{\sqrt{9a^2c^2+4ac-4a^2cr_1+4-8r_1a+4r_1^2a^2}}{c(-1+r_1a)}\right] \tag{12}$$

- Substituting the above result in (8) give us respectivly

$$R_1^{(2)} = \frac{11}{4}a + \frac{1}{4}(11r_1-16c)a^2 + \frac{1}{16}(-128cr_1+44r_1^2+67c^2)a^3 + \ldots \tag{13}$$

$$R_2^{(2)} = a + r_1a^2 + (r_1^2 - \frac{c^2}{4})a^3 + \ldots \tag{14}$$

From two expression for $\bar{a}$ only one with $+$ sign of square root is accepted.( This expression will produced the corrected phrase for the first and second term in series expansion of $R^{(2)}$).
The predicted term is $(r_1^2 - \frac{c^2}{4})$ or $(r_1^2 - \frac{\beta_1}{4\beta_0})$ which is unique.

## NNLO predicted term

- In this case, the observable $R$ is written as:

$$R^{(3)} = a(1 + r_1 a + r_2 a^2) \; . \tag{15}$$

For QCD-$\beta$ function we have

$$\frac{\partial a}{\partial \tau} = \hat{\beta}^3(a) = -a^2(1 + ca + c_2 a^2) \; . \tag{16}$$

Coupling constant $a$ is also satisfies $\frac{\partial a}{\partial c_2} = {\beta_2}^{(3)}(a)$ where ${\beta_2}^{(3)}(a)$ is the third order approximation to the $\beta_i$ which is defined by:

$$\beta_i = -\hat{\beta}(a) \int_0^a \frac{x^{i+2}}{[\hat{\beta}(a)]^2} \; . \tag{17}$$

- Using the self-consistency principle, lead us to:

$$\begin{aligned} \frac{\partial r_1}{\partial \tau} &= 1 \; , \quad \frac{\partial r_2}{\partial \tau} = c + 2r_1 \; , \\ \frac{\partial r_1}{\partial c_2} &= 0 \; , \quad \frac{\partial r_2}{\partial c_2} = -1 \; . \end{aligned} \tag{18}$$

Solve these set of equations:

$$\begin{aligned} \rho_1 &= \tau - r_1 \; , \\ \rho_2 &= r_2 + c_2 - (r_1 + \frac{c}{2})^2 \; , \end{aligned} \tag{19}$$

which are RS invariant.

- PMS criterion:

$$\frac{\partial R^{(3)}}{\partial \tau} = {\beta_2}^{(3)}(a)[1 + 2r_1 a + 3r_2 a^2]a^2$$
$$+[1 + (c + 2r_1)a]a^2 = 0 \ , \qquad (20)$$
$$\frac{\partial R^{(3)}}{\partial c_2} = {\beta_2}^{(3)}(a)(1 + 2r_1 a + 3r_2 a^2) - a^2 = 0 \qquad (21)$$

Consequently

$$(\bar{c_2} + 2\bar{r_1}c + 3\bar{r_2}) + (2\bar{r_1}\bar{c_2} + 3\bar{c_2})\bar{a} + (3\bar{r_2}\bar{c_2})\bar{a}^2 = 0 \ . \qquad (22)$$

$$\int_0^{\bar{a}} \frac{dx}{(1 + cx + \bar{c}_2 x^2)^2} = \frac{\bar{a}}{1 + (c + 2\bar{r}_1)\bar{a}} \ . \qquad (23)$$

Doing the integral in Eq.(23) and expanding the result up to $O(\bar{a}^3)$

$$\int_0^{\bar{a}} \frac{dx}{(1 + cx + \bar{c}_2 x^2)^2} = \bar{a} - c\bar{a}^2 + (c^2 - \frac{2}{3}\bar{c}_2)\bar{a}^3 + \dots \ . \qquad (24)$$

Equating above result to the right hand side of Eq.(23), we will obtain

$$\bar{r}_1 = -\frac{1}{2}\,\frac{\bar{a}(3c^3\bar{a} - 2\bar{c}_2 - 2\bar{c}_2 c\bar{a})}{3 - 3c\bar{a} + 3c^2\bar{a}^2 - 2\bar{c}_2\bar{a}^2} \ . \qquad (25)$$

Eq.(22) and using Eq.(25):

$$\bar{r}_2 = \frac{(-3\bar{c}_2 + \bar{c}_2 c\bar{a} - 5\bar{c}_2 c^2\bar{a}^2 + 3c^4\bar{a}^2 + 3\bar{c}_2 c^3\bar{a}^3 - 2\bar{c}_2^2 c\bar{a}^3)}{-3(3 - 3c\bar{a} + 3c^2\bar{a}^2 - 2\bar{c}_2\bar{a}^2)(1 + c\bar{a} + \bar{c}_2\bar{a}^2)} \ . \qquad (26)$$

- All that remains is to find $\bar{a}$ and $\bar{c}_2$ in terms of $a$, $c_2$ and $r_1$ and $r_2$. Rewriting $\rho_1$ and $\rho_2$ in two different scales, will give us:

$$\frac{1}{a} - r_1 - (\frac{1}{\bar{a}} - \bar{r}_1) = 0 \tag{27}$$

$$r_2 + c_2 - (r_1 + \frac{c}{2})^2 - (\bar{r}_2 + \bar{c}_2 - (\bar{r}_1 + \frac{c}{2})^2) = 0 \tag{28}$$

By substituting the expression for $\bar{r}_1$ and $\bar{r}_2$ in the set of Eq.(27) and Eq.(28), we are able to find $\bar{a}$ and $\bar{c_2}$ in terms of $a$, $r_1$, $r_2$ and $c_2$.

- Final stage is to substitute the result for $\bar{r}_1$, $\bar{r}_2$ and $\bar{a}$ in the expression for the optimized $R^{(3)}_{opt}$:

$$R^{(3)}_{opt} = \bar{a}(1 + \bar{r}_1\bar{a} + \bar{r}_2\bar{a}^2); \tag{29}$$

By expanding the above equation in terms of $a$, the predicted term for $r_3$ can be extracted.

Since the set equations (27) and (28) are of order 6 and 3 with respect to $\bar{a}$ and $\bar{c}_2$, it seems that the predicted term is not unique.

## Higher order predicated terms

The strategy to predict higher order terms is now obvious:

1) Our desired observale $R$ has a perturbative series as:

$$R^{(k+1)} = a(1 + r_1 a + r_2 a^2 + ... + r_k a^k) \ . \tag{30}$$

2) QCD $\beta$-funcion will be appeared in the following form:

$$\frac{\partial a}{\partial \tau} = \hat{\beta}^{(k+1)} = -a^2(1 + ca + c_2 a^2 + ... + c_k a^k) \ . \tag{31}$$

3) Self-consistency principle will be:

$$\frac{\partial R^{(k+1)}}{\partial(\tau, c_2, ..., c_k)} = O(a^{(k+2)}) \ . \tag{32}$$

Reminding: The dependence of coupling constant $a$ to $c_i$ parameter is given by Eq.(17):$\beta_i = \frac{\partial a}{\partial c_i} = -\hat{\beta}(a) \int_0^a \frac{x^{i+2}}{[\hat{\beta}(a)]^2}$. Using this principle, we will arrive at the following partial differentials:

$$\frac{\partial r_l}{\partial \tau} = \sum_{m=0}^{l-1} (m+1) r_m c_{\,l-m-1} \tag{33}$$

$$\frac{\partial r_l}{\partial c_j} = \begin{cases} \frac{-1}{j-1} \sum_{m=0}^{l-j} r_m W^j_{l-j-m}, & l \geq j \\ \\ 0, & l < j \end{cases} , \tag{34}$$

where $c_0 = r_0 = W^j_0 = 1$ and $c_1 = c$.

The $W_n^j$ are the expansion coefficients of the $\beta_i = \frac{\partial a}{\partial c_i}$ as:

$$\beta_i = \frac{1}{i+1} a^{i+1} (1 + W_1^i a + W_1^2 a^2 + ...) \ . \tag{35}$$

Invariants quantities:

$$\begin{aligned} \rho_1 &= \tau - r_1 \ , \\ \rho_k &= r_k + \frac{N}{k-1} c_k - \Omega^{(k)} \ , \end{aligned} \tag{36}$$

where for instance

$$\begin{aligned} \Omega^{(2)} &= (r_1 + \frac{c}{2})^2 \\ \Omega^{(3)} &= r_1 (c_2 + 3r_2 - 3r_1^2 - \frac{c}{2} r_1) \\ &. \\ &. \\ &. \end{aligned} \tag{37}$$

4) Employing PMS criterion lead us to:

$$\frac{\partial R^{(i)}}{\partial (a, c_2, ..., c_k)}_{|a=\bar{a},\ c_2=\overline{c_2},\ c_3=\overline{c_3},\ ...c_k=\overline{c_k}} = 0 \ , \tag{38}$$

which leads us to

$$\sum_{l=0}^{k} a^l \sum_{m=l}^{k} (1+m) r_m c_{k+l-m} = 0$$

$$\int_0^{\bar{a}} \frac{x^{j+2}}{[\hat{\beta}^{(k+1)}]^2} = \frac{\bar{a}^{j-1}}{(j-1)} \frac{[\sum_{l=0}^{k-j} a^l \sum_{m=0}^{l} (1+m) r_m W_{l-m}^j]}{[\sum_{l=0}^{k-1} a^l \sum_{m=0}^{l} (1+m) r_m c_{l-m}]} \ . \tag{39}$$

5) Extracting predicted higher order terms:

a) Equations (39) give us $\bar{r_1}$, $\bar{r_2}$, $\bar{r_3}$,..., $\bar{r_k}$ in terms of $\bar{a}$, $\bar{c_2}$, $\bar{c_3}$, ...., $\bar{c_k}$.

b) Using $\rho_1$, $\rho_2$, $\rho_3$ and ... in two different scales, it is possible to find $\bar{a}$, $\bar{c_2}$, $\bar{c_3}$..., $\bar{r_k}$ in terms of $a$, $r_1$, $r_2$, $r_3$, ...$r_k$, $c_2$, $c_3$, ... $c_k$.

c) Substituting all the results in the optimized expression for $R_{opt}$

$$R_{opt}^{(k+1)} = \bar{a}(1 + \bar{r}_1\bar{a} + \bar{r}_2\bar{a}^2 + \bar{r}_3\bar{a}^3 + ...\bar{r}_k\bar{a}^k) \qquad (40)$$

and expanding the final result in term of coupling constant $a$, the required predicted term will be obtained.

*It is seems that as before we will not get a unique prediction.*

## Compete Renormalization Group Improvement

- An observable $R(Q)$ in a standard approach:

$$R(Q) = a + r_1 a^2 + r_2 a^3 + \cdots + r_n a^{n+1} + \cdots . \quad (41)$$

In new approach:

$$R(Q) = a_0 + X_2 a_0^3 + X_3 a_0^4 + \cdots + X_n a_0^{n+1} + \cdots . \quad (42)$$

- In Eq. (41) all terms depend on renormalization scale ($\mu$), while in Eq. (42), $a_0 = a_0(Q)$. $X_2, X_3, \cdots$ are constants and scheme invariants( before $\rho_2$ and $\rho_3$).
- Self consistency principle + solving simultaneously the related partial differential equations:

$$\begin{aligned} r_2(r_1, c_2) &= r_1{}^2 + c r_1 + X_2 - c_2 \\ r_3(r_1, c_2, c_3) &= r_1{}^3 + \frac{5}{2} c r_1{}^2 + (3X_2 - 2c_2) r_1 + X_3 - \frac{1}{2} c_3 \\ \vdots \quad &\quad \vdots . \end{aligned} \quad (43)$$

In general the structure is

$$r_n(r_1, c_2, \ldots, c_n) = \hat{r}_n(r_1, c_2, \ldots, c_{n-1}) + X_n - c_n/(n-1) . \quad (44)$$

- The coupling constant $a_0$ represents a summation over NLO contribution of all terms in Eq. (41) which is an RS independent sum. It is defined as:

$$a_0 \equiv a + r_1 a^2 + (r_1^2 + c r_1 - c_2) a^3 + (r_1^3 + \frac{5}{2} c r_1^2 - 2 c_2 r_1 - \frac{1}{2} c_3) a^4 + \ldots . \quad (45)$$

## Predicted terms in CORGI

- a) NLO approximation:

$$R(Q) = a_0 \tag{46}$$

Substituting Eq.(45):

$$R(Q) = a + r_1 a^2 + (r_1^2 + cr_1 - c_2)a^3 + \ldots . \tag{47}$$

Predictd term:

$$r_2(pre) = r_1^2 + cr_1 - c_2 \text{ or } r_2(pre) = r_1^2 + \frac{\beta_1}{\beta_0} r_1 - \frac{\beta_2}{\beta_0} \tag{48}$$

b) NNLO approximation:

$$R(Q) = a_0 + X_2 a_0^3 \tag{49}$$

Substituting Eq.(45) for $a_0$ and the related expression for $X_2$ (Eq.(43)) and rearrange them in terms of $a$, we will obtain

$$R(Q) = a + r_1 a^2 + r_2 a^3 + \left( r1^3 + \frac{5}{2} c r_1^2 - 2c_2 r_1 - \frac{1}{2} c_3 + 3(r_2 - r_1^2 - cr_1 + c_2) r_1 \right) a^4 \tag{50}$$

Predictd term is:

$$r_3(pre) = r1^3 + \frac{5}{2} \frac{\beta_1}{\beta_0} r_1^2 - 2\frac{\beta_2}{\beta_0} r_1 - \frac{1}{2} \frac{\beta_3}{\beta_0} + 3(r_2 - r_1^2 - \frac{\beta_1}{\beta_0} r_1 + \frac{\beta_2}{\beta_0}) r_1 \tag{51}$$

This procedure can be extended to predice higher oreder terms. The results are unique.