K_{l3} Form Factor with $N_f = 2+1$ Domain Wall Fermions

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UKQCD/RBC Collaborations

Acknowledgements

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CKM Matrix

Quark Mixing Matrix

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

Probability of a transition from one quark q to another quark q'

$$\propto |V_{qq'}|^2$$

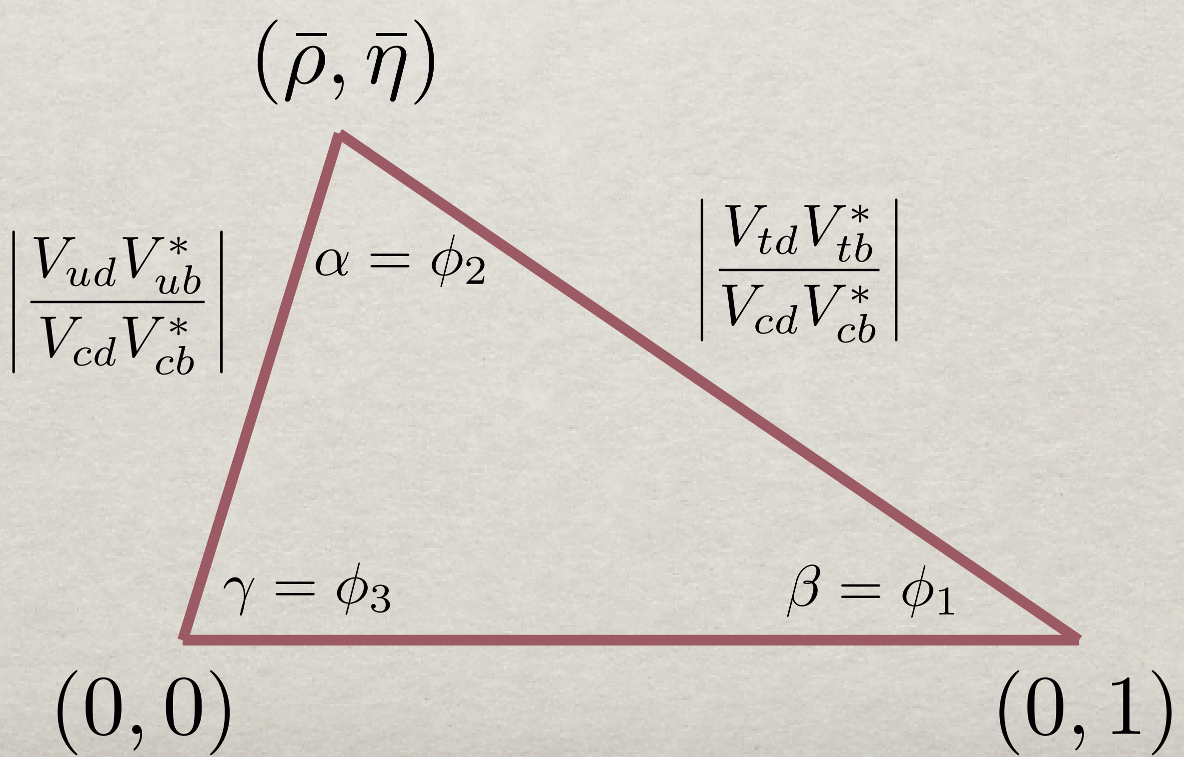
$$V_{CKM} \begin{bmatrix} |d\rangle \\ |s\rangle \\ |b\rangle \end{bmatrix} = \begin{bmatrix} |d'\rangle \\ |s'\rangle \\ |b'\rangle \end{bmatrix}$$

Unitarity Triangle

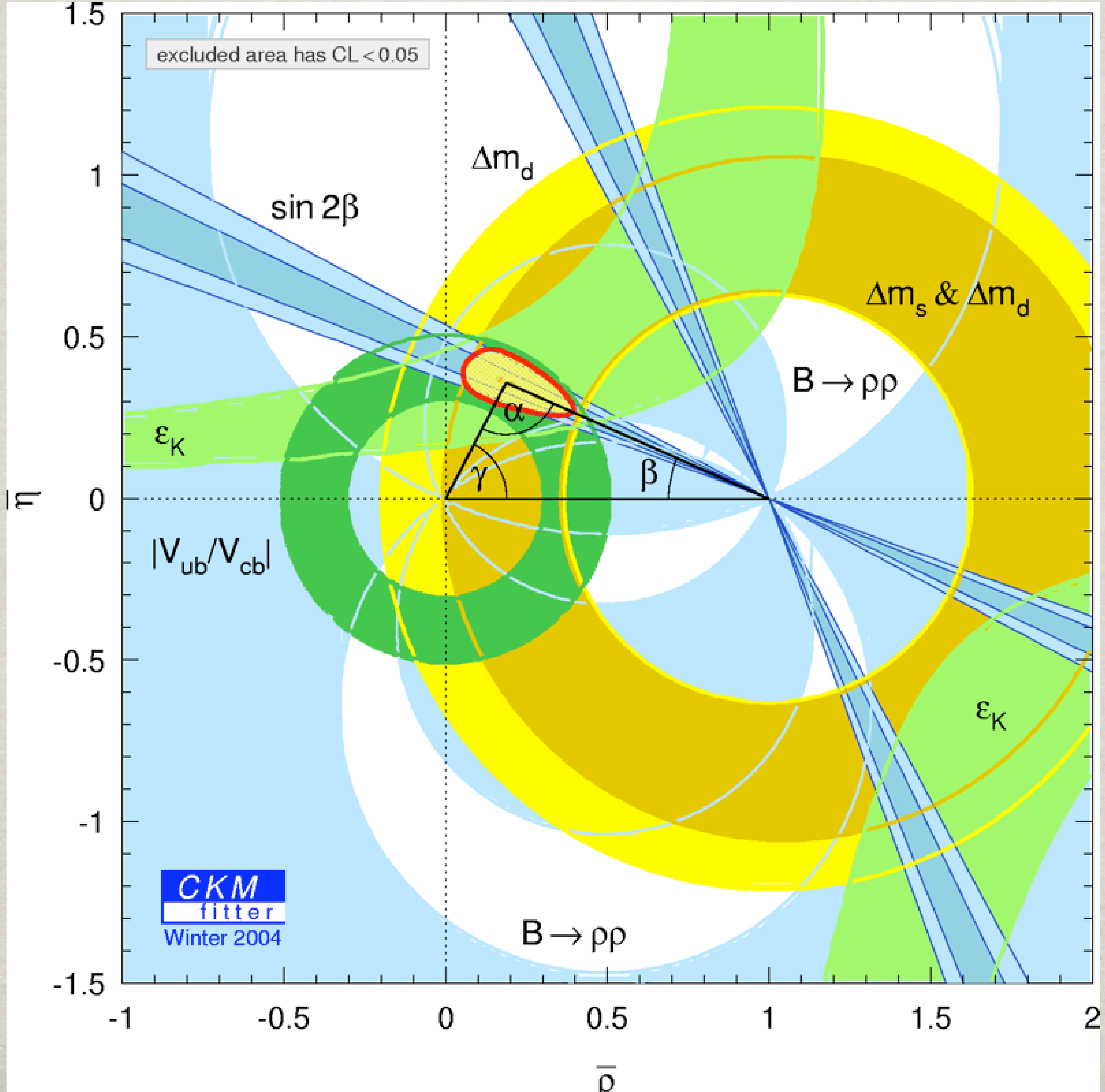
One common parameterisation (Wolfenstein):

$$V = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

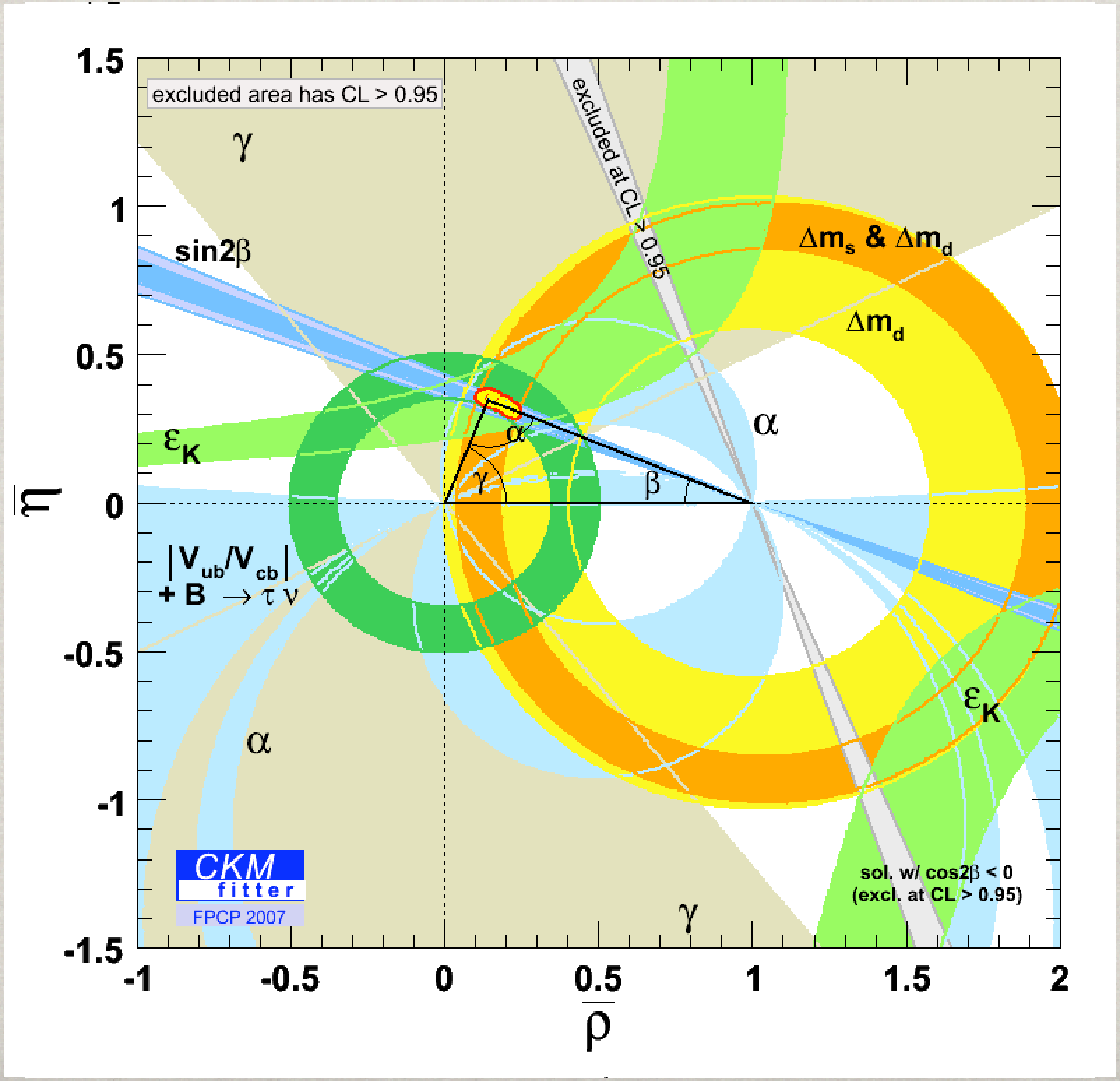
Unitarity: $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$



Unitarity Triangle



Unitarity Triangle



CKM Matrix

Lattice Input

$|V_{us}|$

$$K^+ \to \pi^0 l^+ \nu,\ f_\pi/f_K,\ \Xi^0 \to \Sigma^+ l^- \nu,\ \Sigma^- \to n\, l^- \nu$$

$|V_{ub}|$

$$B \to \pi l \nu$$

$|V_{cd}|$

$$D \to K l \nu,\ D \to \pi l \nu$$

$|V_{cs}|$

$$D \to K l \nu,\ D \to \pi l \nu,\ f_{D_s}$$

$|V_{td}|$ & $|V_{ts}|$

$$f_{B_d}\sqrt{\hat{B}_{B_d}},\ \xi = (f_{B_s}\sqrt{B_{B_s}})/(f_{B_d}\sqrt{B_{B_d}})$$

CKM Matrix

Lattice Input

This talk

$|V_{us}|$

$$K^+ \to \pi^0 l^+ \nu, \; f_\pi / f_K, \; \Xi^0 \to \Sigma^+ l^- \nu, \; \Sigma^- \to n\, l^- \nu$$

$|V_{ub}|$

$$B \to \pi l \nu$$

$|V_{cd}|$

$$D \to K l \nu, \; D \to \pi l \nu$$

$|V_{cs}|$

$$D \to K l \nu, \; D \to \pi l \nu, \; f_{D_s}$$

$|V_{td}|$ & $|V_{ts}|$

$$f_{B_d} \sqrt{\hat{B}_{B_d}}, \; \xi = (f_{B_s} \sqrt{B_{B_s}}) / (f_{B_d} \sqrt{B_{B_d}})$$

Motivation

- $K \to \pi l \nu \ (K_{l3})$ decay leads to determination of $|V_{us}|$

$$decayrate \propto |V_{us}|^2 |f_+(q^2 = 0)|^2$$

- Require precise theoretical determination $f_+(0)$
- Current conservation ⟹ $f_+(0) = 1\Big|_{su(3)\ flavour\ limit}$
- Ademollo-Gatto Theorem -> second order SU(3) breaking effects in $f_+(0)$

$$f_+(0) = 1 + f_2 + f_4 + \cdots$$
$$\Rightarrow \Delta f = 1 + f_2 - f_+(0)$$

- [Leutwyler & Roos: $f_2 = -0.023$]

Motivation

$$\Delta f = 1 + f_2 - f_+(0)$$

$\Delta f =$

-0.016(8)	(Leutwyler & Roos, 1984)
-0.017(5)(7)	(Bećirević et al., quenched)
-0.009(9)	(Dawson et al., $N_f = 2$ DWF)
-0.025(4)	(Tsutsui et al., $N_f = 2$ Clover)
-0.0161(51)	(Preliminary RBC/UKQCD)

Improve on earlier studies by:

- Using $N_f = 2 + 1$ flavours of dynamical fermions
- Probing light quark masses
- Checking finite size effects

Lattice Techniques

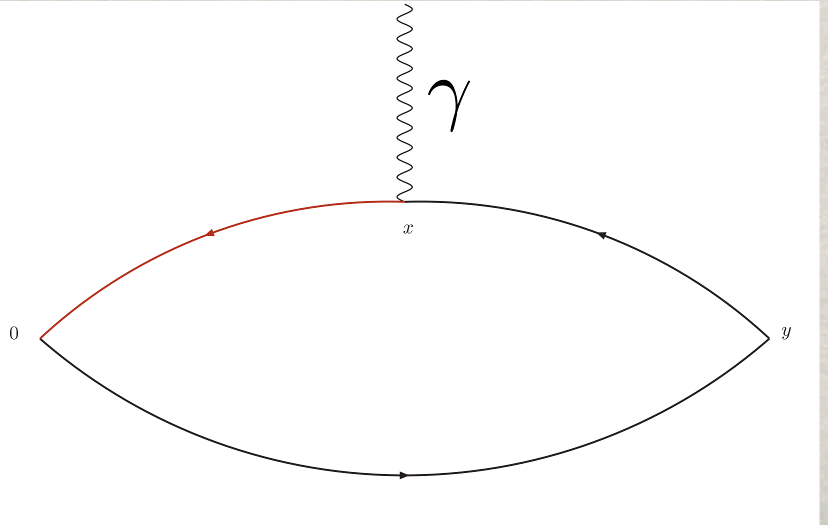
$K \to \pi$ matrix element

$$\langle \pi(p')|V_\mu|K(p)\rangle = (p_\mu + p'_\mu)f_+(q^2) + (p_\mu - p'_\mu)f_-(q^2), \quad q^2 = (p' - p)^2$$

Three-point function

$$C_\mu^{PQ}(t', t, \vec{p}', \vec{p}) = \sum_{\vec{x},\vec{y}} e^{-i\vec{p}'(\vec{y}-\vec{x})} e^{-i\vec{p}\vec{x}} \langle 0|\mathcal{O}_Q(t')|Q(p')\rangle\langle Q(p')|V_\mu(t)|P(p)\rangle\langle P(p)|\mathcal{O}_P^\dagger(0)|0\rangle$$

$K(p)$



$\pi(p')$

Extraction of Form Factor

[hep-ph/0403217,0607162]

Extract scalar form factor

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2)$$

at $q^2_{\rm max} = (m_K - m_\pi)^2$ with high precision via

$$R(t',t) = \frac{C_4^{K\pi}(t',t;\vec{0},\vec{0})C_4^{\pi K}(t',t;\vec{0},\vec{0})}{C_4^{KK}(t',t;\vec{0},\vec{0})C_4^{\pi\pi}(t',t;\vec{0},\vec{0})}$$

$$\longrightarrow \frac{(m_K + m_\pi)^2}{4 m_K m_\pi} |f_0(q^2_{\rm max})|^2$$

Q^2 Dependence

Construct second ratio

$$\tilde{R}(t',t;\vec{p}\,',\vec{p}) = \frac{C_4^{K\pi}(t',t;\vec{p}\,',\vec{p})C^K(t;\vec{0})C^\pi(t'-t;\vec{0})}{C_4^{K\pi}(t',t;\vec{0},\vec{0})C^K(t;\vec{p})C^\pi(t'-t;\vec{p}\,')}$$
$$\longrightarrow \frac{(E_K(\vec{p})+E_\pi(\vec{p}\,'))^2}{m_K+m_\pi}F(p',p)$$

where

$$F(p',p) = \frac{f_+(q^2)}{f_0(q^2_{\mathrm{max}})}\left(1+\frac{E_K(\vec{p})-E_\pi(\vec{p}\,')}{E_K(\vec{p})+E_\pi(\vec{p}\,')}\xi(q^2)\right)\,,\ \xi(q^2) = \frac{f_-(q^2)}{f_+(q^2)}$$

Q^2 Dependence

Construct third ratio

$$R_k(t',t;\vec{p}\,',\vec{p}) = \frac{C_k^{K\pi}(t',t;\vec{p}\,',\vec{p})C_4^{KK}(t',t;\vec{p}\,',\vec{p})}{C_4^{K\pi}(t',t;\vec{p}\,',\vec{p})C_k^{KK}(t',t;\vec{p}\,',\vec{p})} \quad (k=1,2,3)$$

to obtain

$$\xi(q^2) = \frac{-(E_K(\vec{p})+E_K(\vec{p}\,'))(p+p')_k + (E_K(\vec{p})+E_\pi(\vec{p}\,'))(p+p')_k R_k}{(E_K(\vec{p})+E_K(\vec{p}\,'))(p-p')_k - (E_K(\vec{p})-E_\pi(\vec{p}\,'))(p+p')_k R_k}$$

Parameters

- $N_f = 2 + 1$ flavours of dynamical domain wall fermions
- Iwasaki gauge action

$\beta = 2.13,\ \ L_s = 16,\ \ am_{\rm res} \approx 0.003,\ \ a \approx 0.121\,\mathrm{fm},\ \ am_s = 0.04$

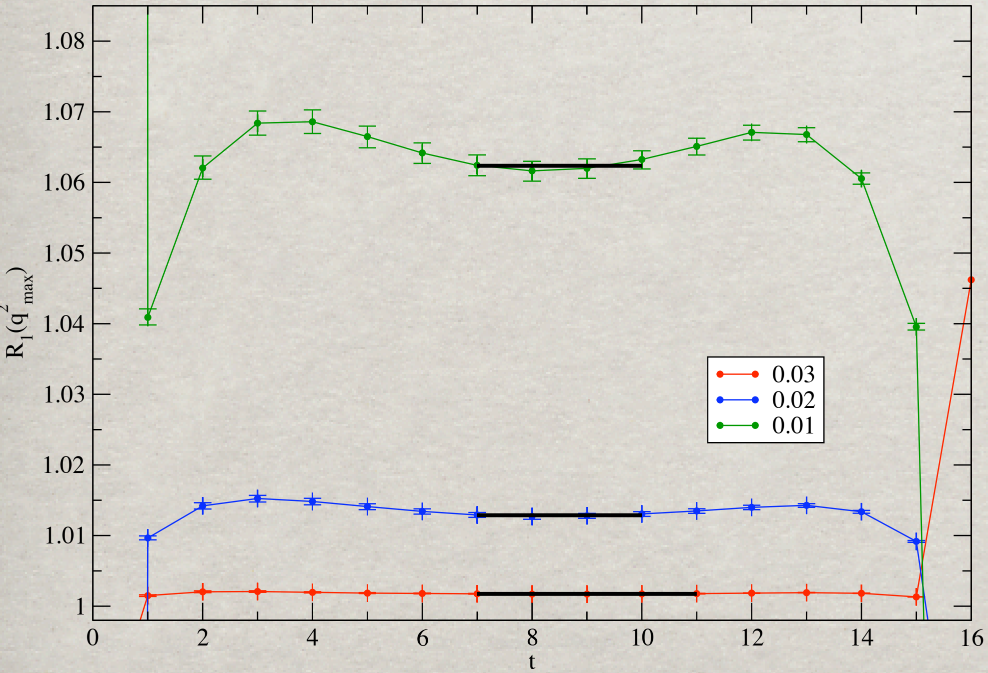
am_q	Volume	m_π [MeV]	m_K [MeV]
0.03	$16^3 \times 32$	0.632(1)	0.677(1)
0.02		0.522(2)	0.624(2)
0.01		0.401(2)	0.575(1)
0.03	$24^3 \times 64$	0.628(1)	0.673(1)
0.02		0.521(1)	0.621(1)
0.01		0.390(1)	0.566(1)
0.005		0.308(1)	0.539(1)

For more details, see hep-lat/0701013

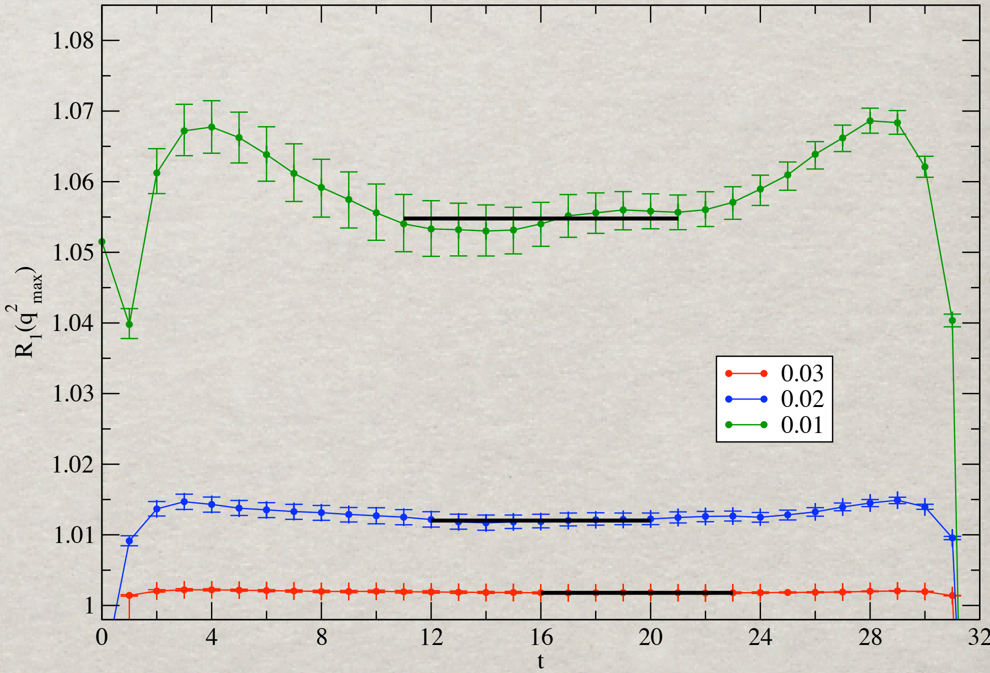
$f_0(q^2_{\max}),\ am_s = 0.04$

$$R(t',t) = \frac{C_4^{K\pi}(t',t;\vec{0},\vec{0})C_4^{\pi K}(t',t;\vec{0},\vec{0})}{C_4^{KK}(t',t;\vec{0},\vec{0})C_4^{\pi\pi}(t',t;\vec{0},\vec{0})}$$

$16^3 \times 32$



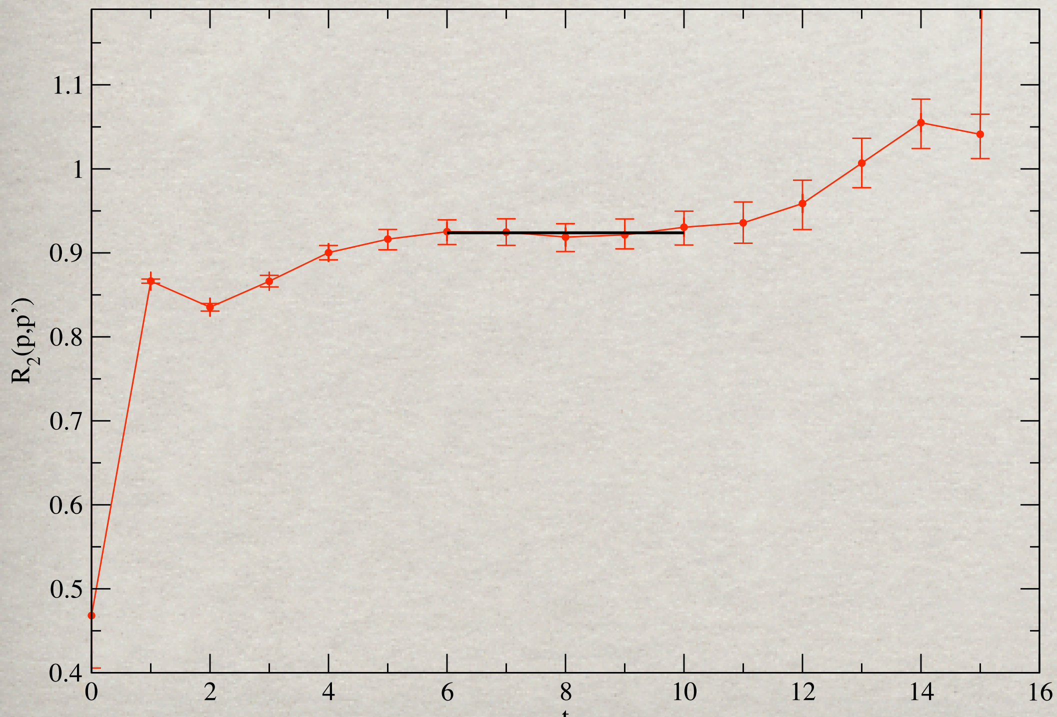
$24^3 \times 64$



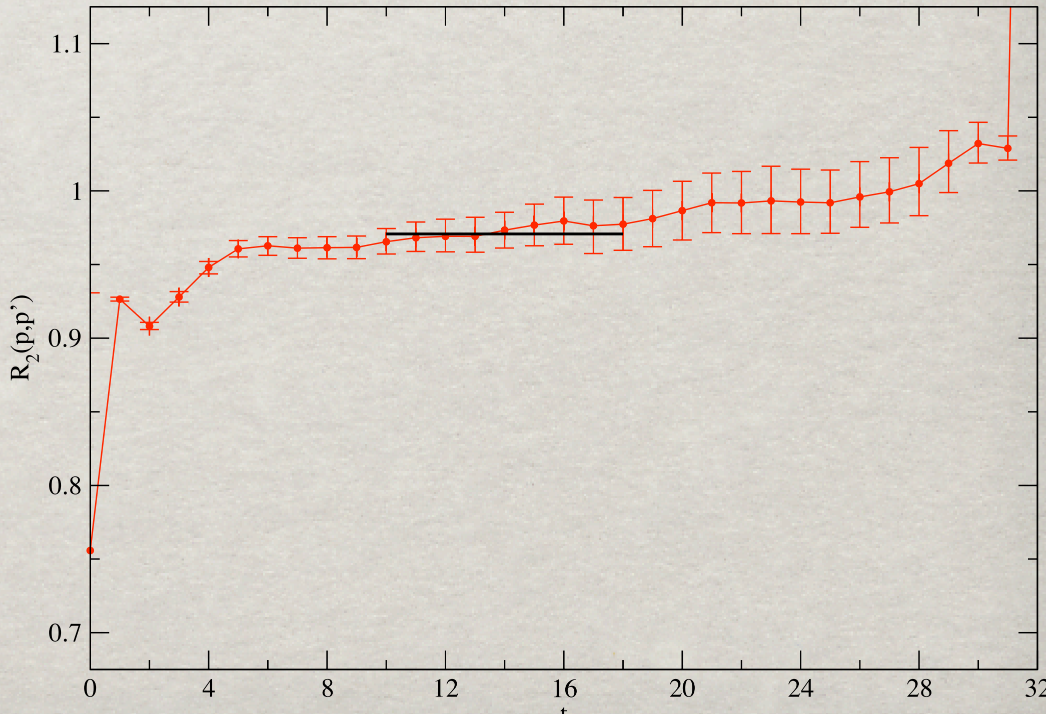
$$F(p,p'),\ am_q = 0.02,\ |\vec{q}|^2 = 1:$$

$$\tilde{R}(t',t;\vec{p}\,',\vec{p}) = \frac{C_4^{K\pi}(t',t;\vec{p}\,',\vec{p})C^K(t;\vec{0})C^\pi(t'-t;\vec{0})}{C_4^{K\pi}(t',t;\vec{0},\vec{0})C^K(t;\vec{p})C^\pi(t'-t;\vec{p}\,')}$$

$16^3 \times 32$



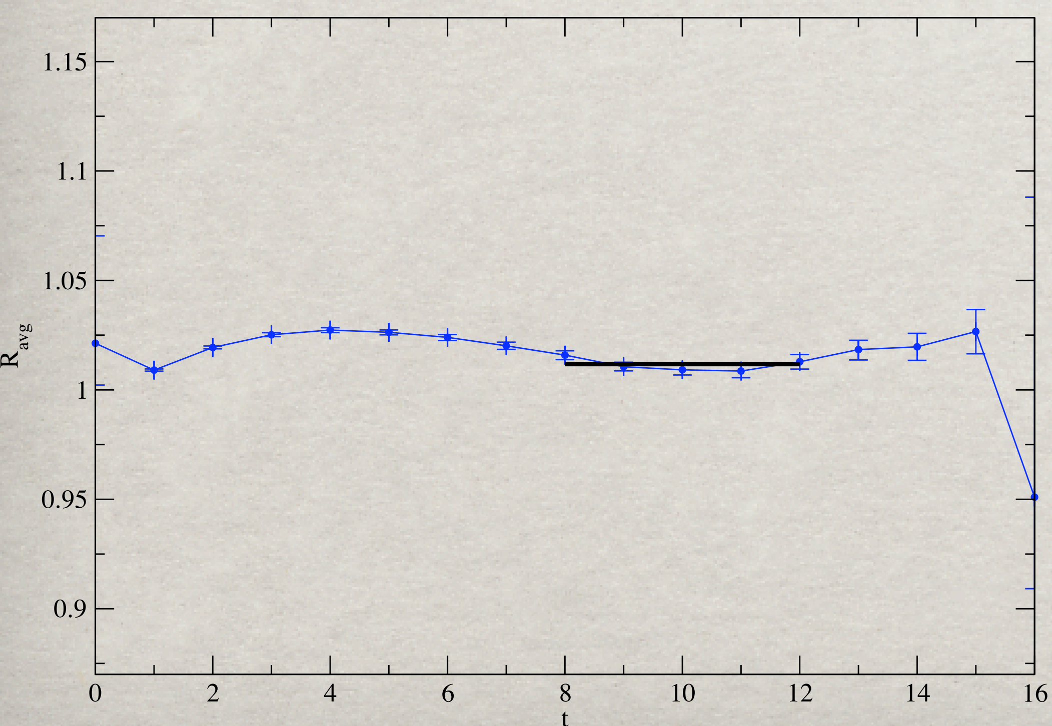
$24^3 \times 64$



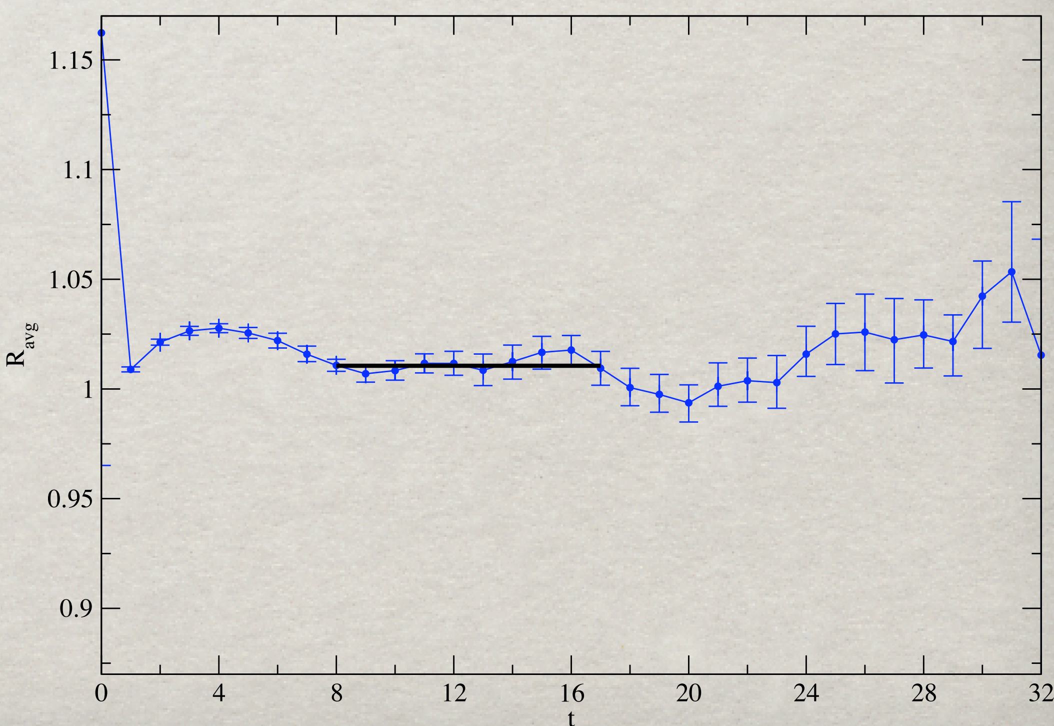
$$\xi(q^2),\ \ am_q = 0.03,\ \ |\vec{q}|^2 = 2:$$

$$R_k(t',t;\vec{p}\,',\vec{p}) = \frac{C_k^{K\pi}(t',t;\vec{p}\,',\vec{p})\,C_4^{KK}(t',t;\vec{p}\,',\vec{p})}{C_4^{K\pi}(t',t;\vec{p}\,',\vec{p})\,C_k^{KK}(t',t;\vec{p}\,',\vec{p})} \qquad (k=1,2,3)$$

$16^3 \times 32$



$24^3 \times 64$



Fitting Form Factors

- Construct scalar form factor:

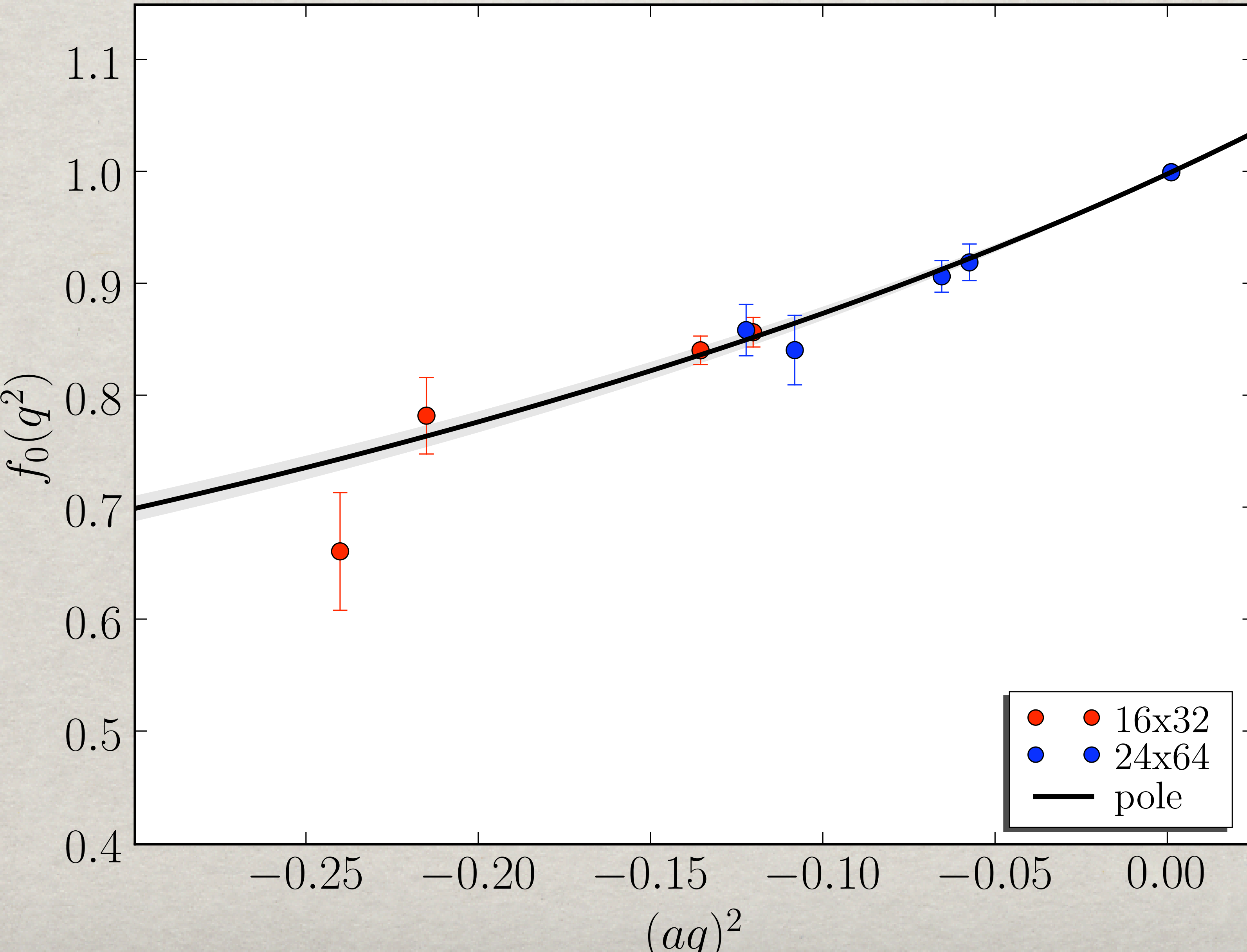
$$f_0(q^2) = f_+(q^2)\Big[1 + \frac{q^2}{m_K^2 - m_\pi^2}\xi(q^2)\Big]$$

- Fit with a monopole ansatz:

$$f_0(q^2) = \frac{f_0(0)}{1 - q^2/M^2}$$

$f_0(q^2)$

$am_s = 0.04,\ am_{ud} = 0.03,\ V = 16^3 \times 32\ \&\ 24^3 \times 64,\ L_s = 16$

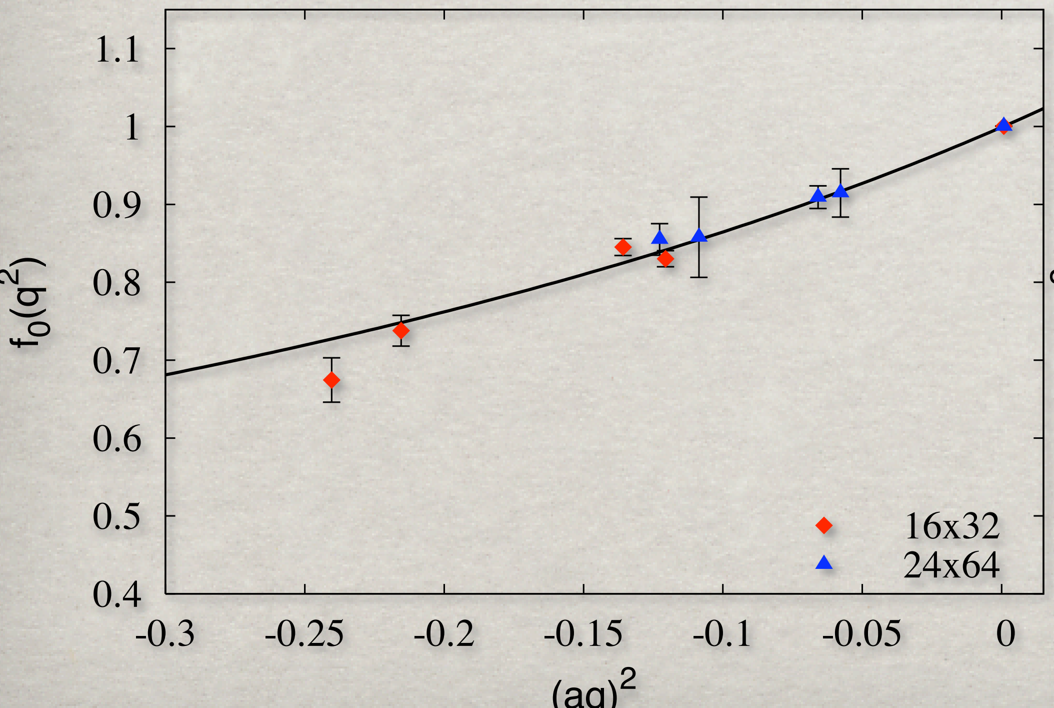


$f_0(q^2_{\mathrm{max}})$

$am_s = 0.04,\ am_{ud} = 0.03,\ V = 16^3 \times 32\ \&\ 24^3 \times 64,\ L_s = 16$

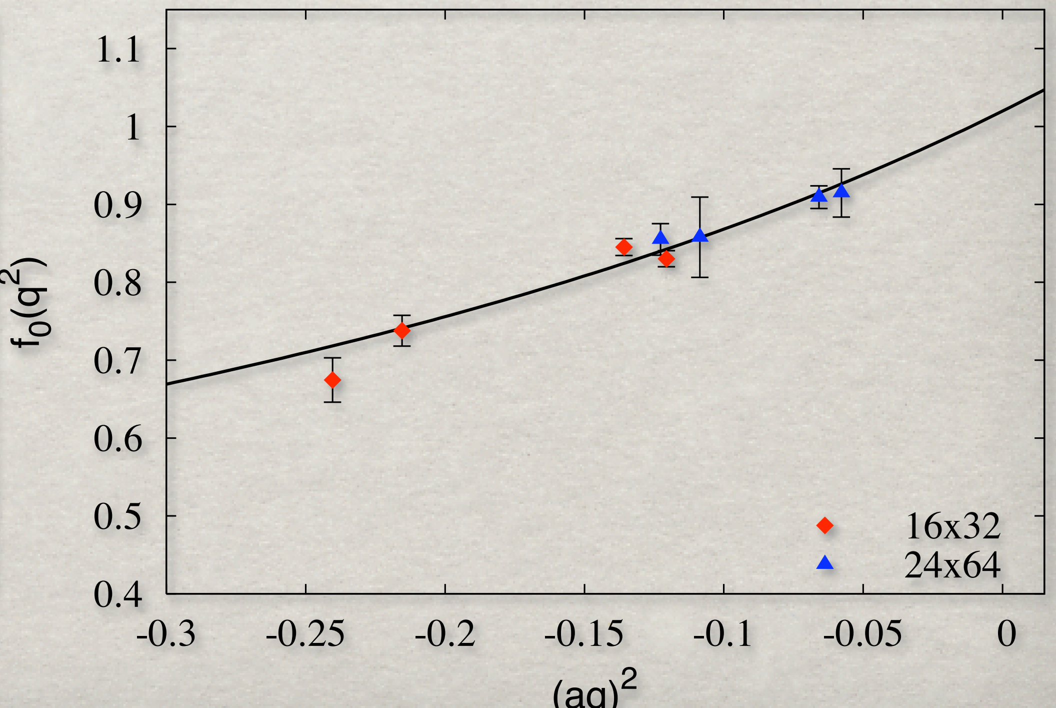
with

$f_0(q^2_{\mathrm{max}}):\ \ f_0(0) = 0.99911(6)$

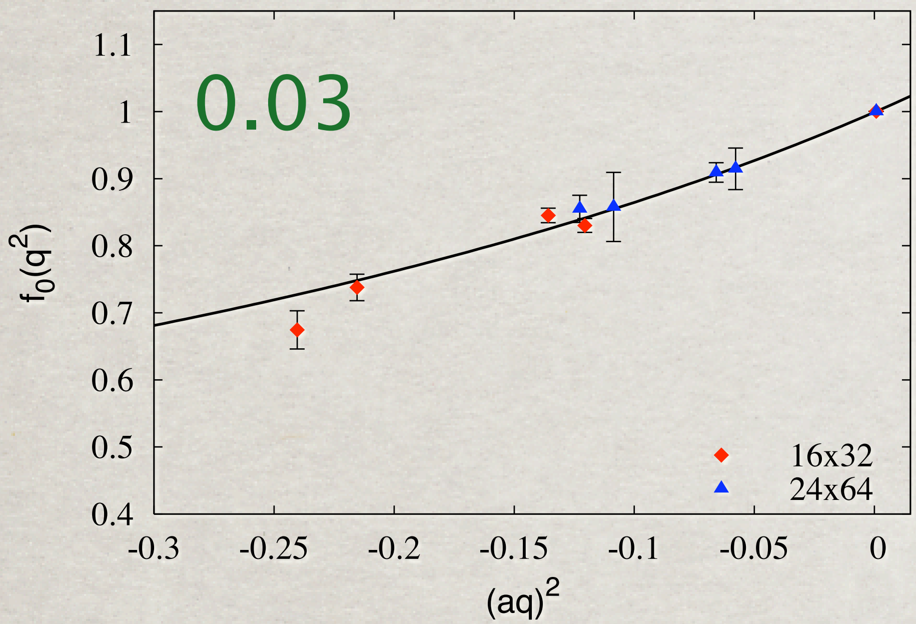


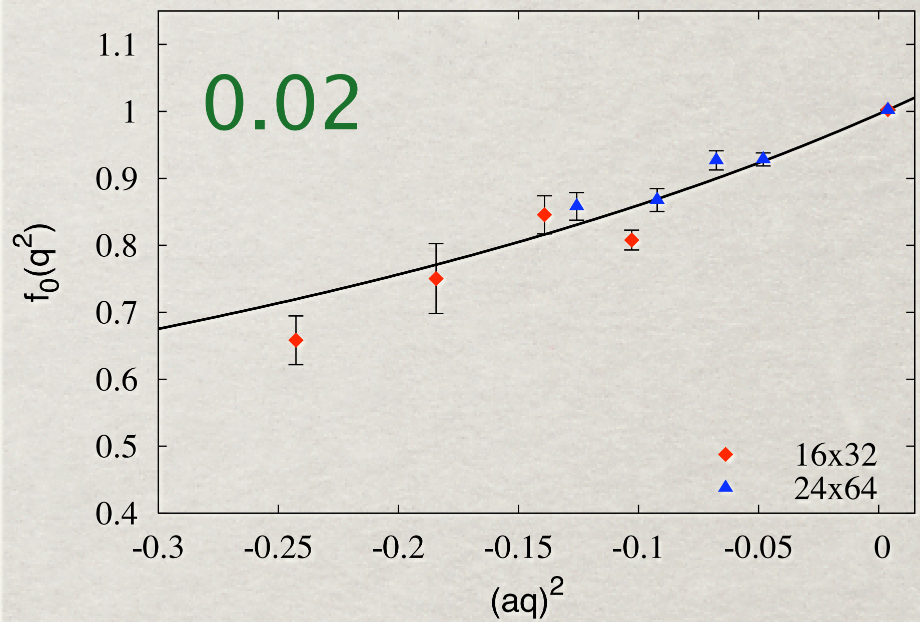
without

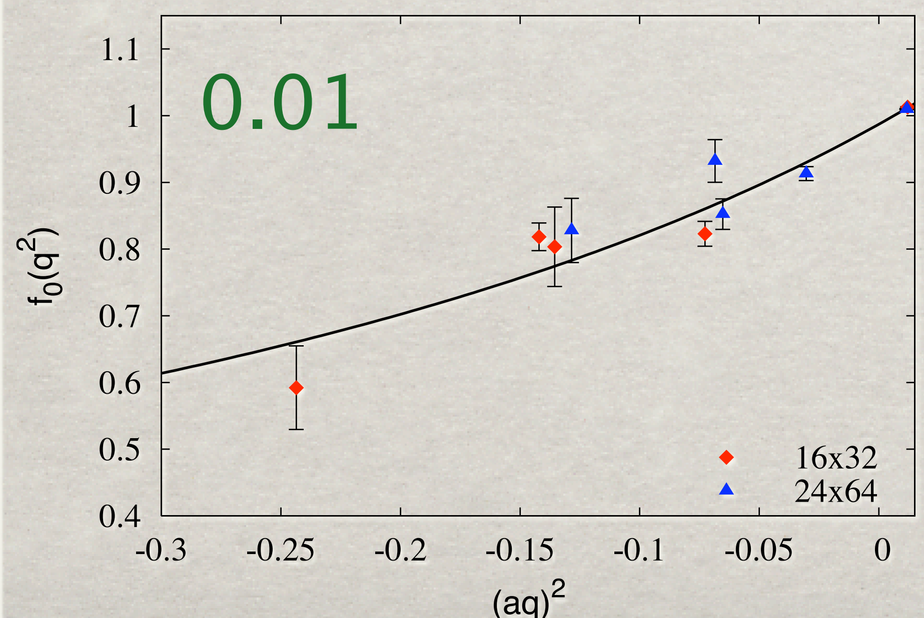
$f_0(q^2_{\mathrm{max}}):\ \ f_0(0) = 1.0198(301)$



Pole: $f_0(q^2) = f_0(0)/(1 - q^2/M^2)$







- 0.03: $f_0(0) = 0.99911(6)$
- 0.02: $f_0(0) = 0.99622(51)$
- 0.01: $f_0(0) = 0.98725(272)$

Chiral Extrapolation of $f_+(0)$

$$f_+(0) = 1 + f_2 + \Delta f$$

$$f_2 = \frac{3}{2} H_{\pi K} + \frac{3}{2} H_{\eta K}$$

where

$$H_{PQ} = -\frac{1}{64\pi^2 f_\pi^2}\left[M_P^2 + M_Q^2 + \frac{2M_P^2 M_Q^2}{M_P^2 - M_Q^2}\log\left(\frac{M_Q^2}{M_P^2}\right)\right]$$

at the physical masses, $f_2 = -0.023$

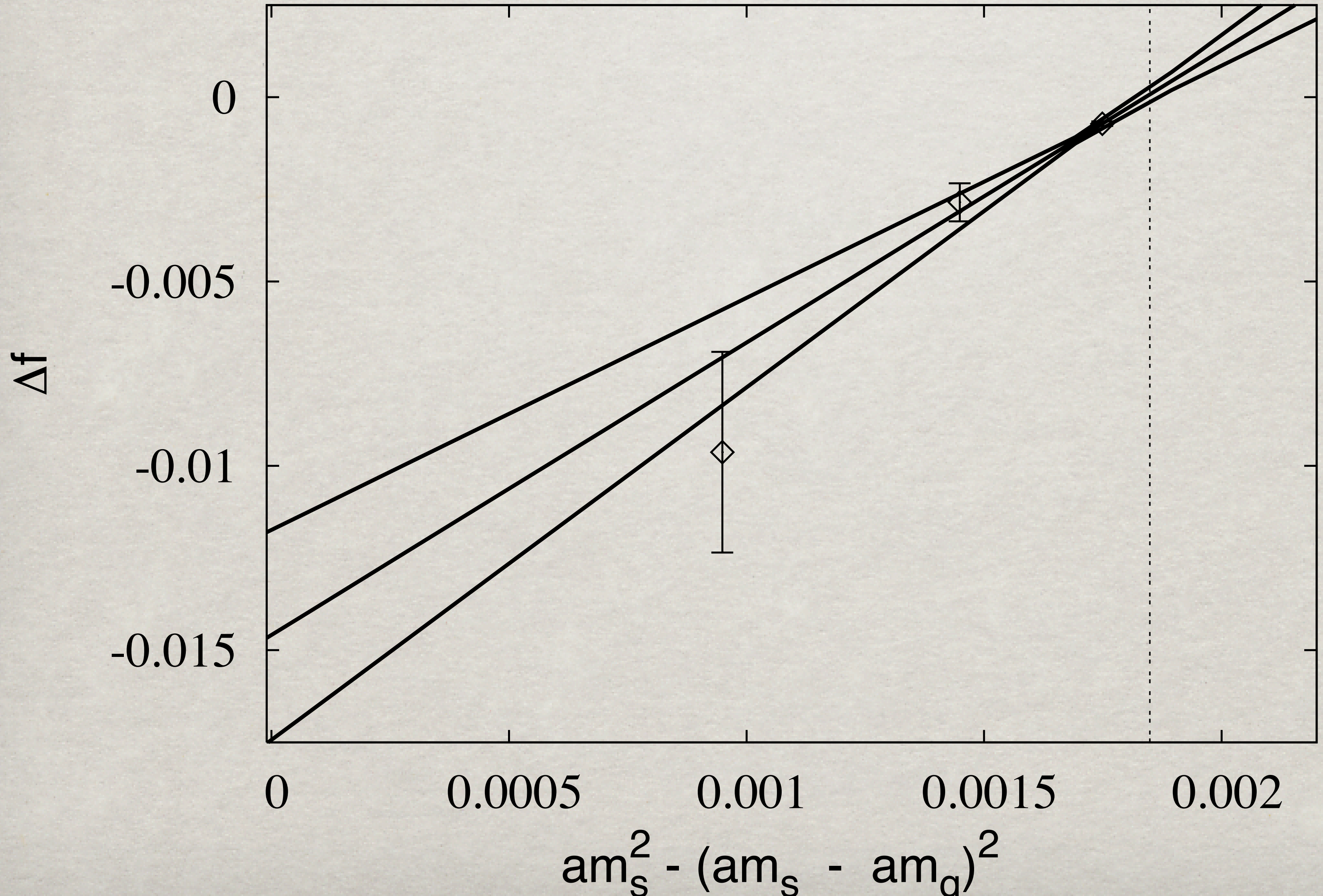
$\Delta f \propto (m_s - m_{ud})^2$ → Attempt two different extrapolations

$$\Delta f = a + B(m_s - m_{ud})^2$$

$$R_{\Delta f} = \frac{\Delta f}{(M_K^2 - M_\pi^2)^2} = a + b(M_K^2 + M_\pi^2)$$

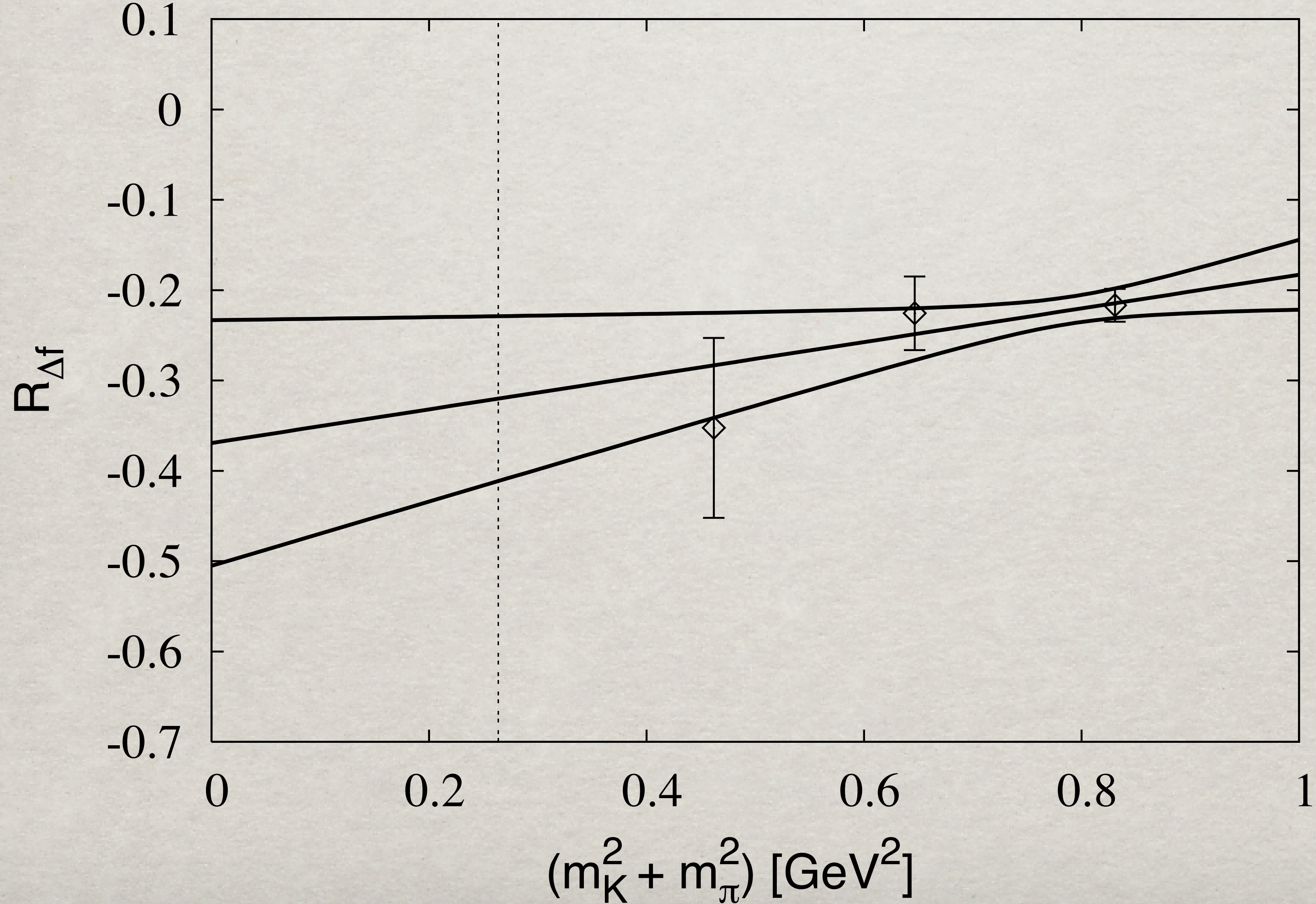
Chiral Extrapolation of $f_+(0)$

$\Delta f = a + B(m_s - m_{ud})^2$ $\Delta f = -0.0146(28)$



Chiral Extrapolation of $f_+(0)$

$$R_{\Delta f} = \frac{\Delta f}{(M_K^2 - M_\pi^2)^2} = a + b(M_K^2 + M_\pi^2) \quad \Delta f = -0.0161(46)$$



Alternative Fits To $f_0(q^2)$

1. linear:

$$f_0(q^2) = f_0(0) + a_1 q^2$$

2. quadratic:

$$f_0(q^2) = f_0(0) + a_1 q^2 + a_2 q^4$$

3. z-fit [hep-ph/0607108]:

$$f_0(t) = \frac{1}{\phi(t, t_0, Q^2)} \sum_{k=0}^{\infty} a_k(t_0, Q^2) z(t, t_0)^k$$

$$t_\pm \equiv (m_K \pm m_\pi)^2$$

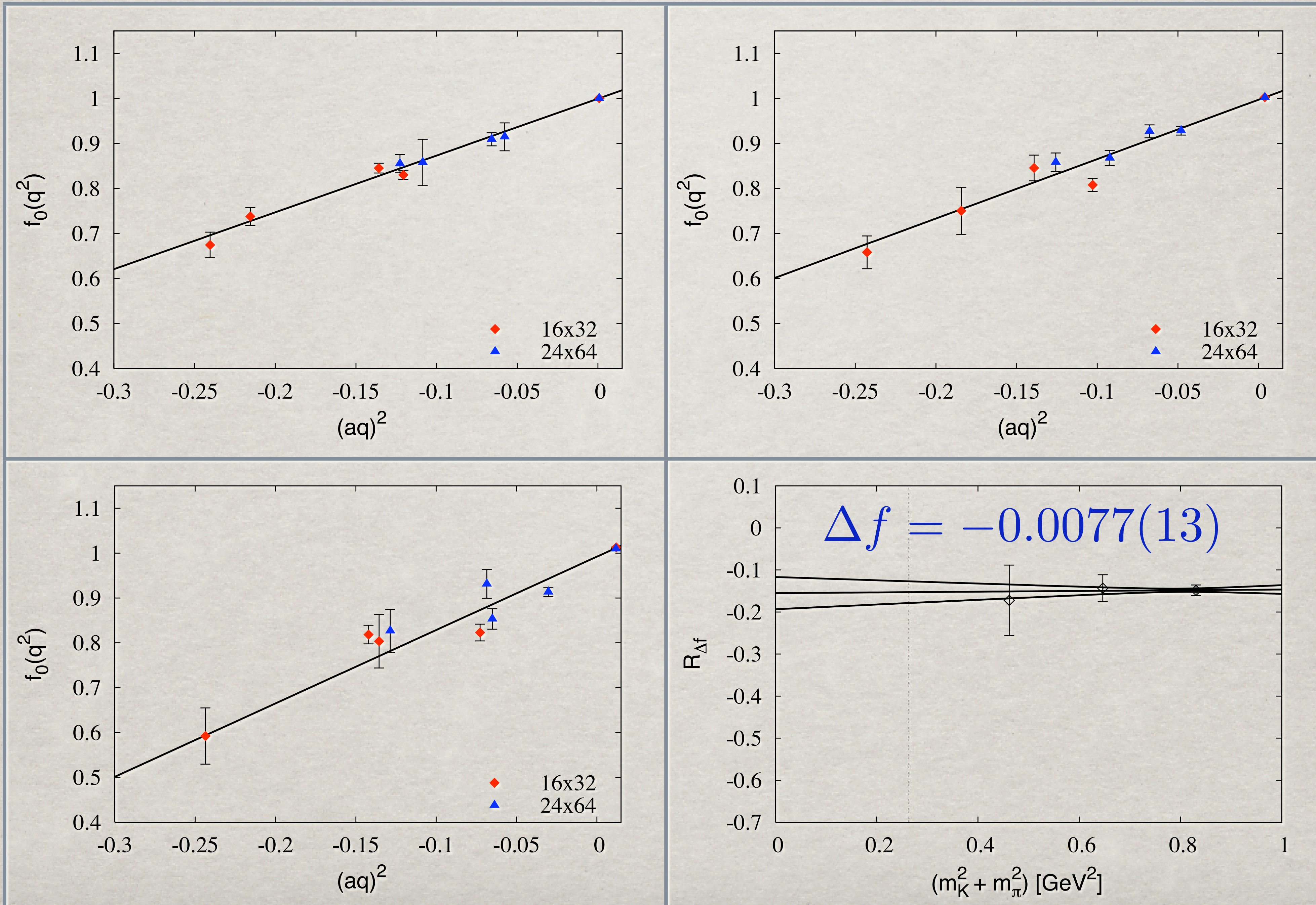
$$t = q^2 \to z(t, t_0) \equiv \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}}$$

$$t_0 \in (-\infty, t_+)$$

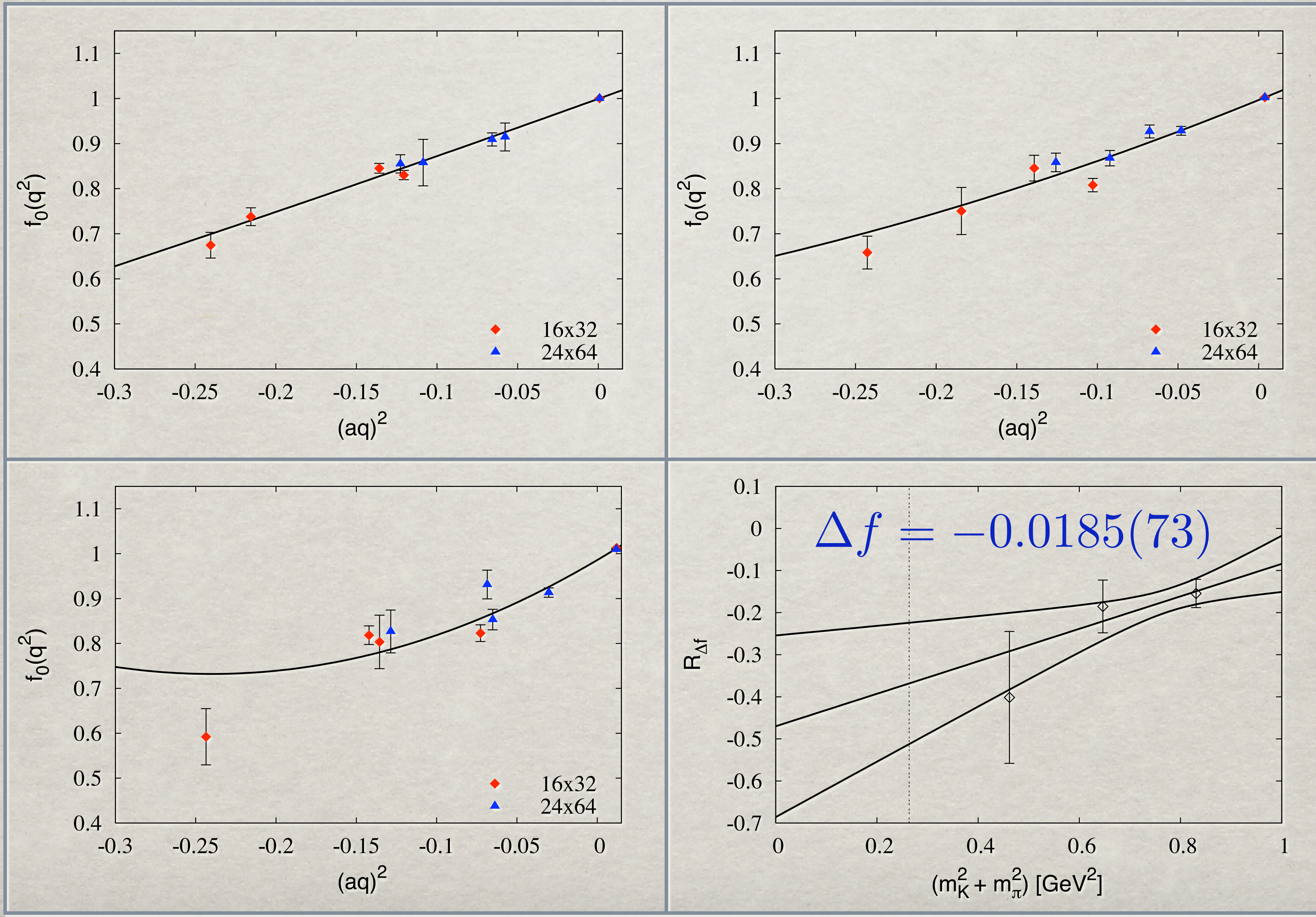
$$t_0 = t_+(1 - \sqrt{1 - t_-/t_+})$$

$$\phi(t, t_0, Q^2) = \sqrt{\frac{3 t_+ t_-}{32\pi}} \frac{z(t,0)}{-t} \frac{z(t, -Q^2)}{-Q^2 - t} \left(\frac{z(t, t_0)}{t_0 - t}\right)^{-1/2} \left(\frac{z(t, t_-)}{t_- - t}\right)^{-1/4} \frac{\sqrt{t_+ - t}}{(t_+ - t_0)^{1/4}}$$

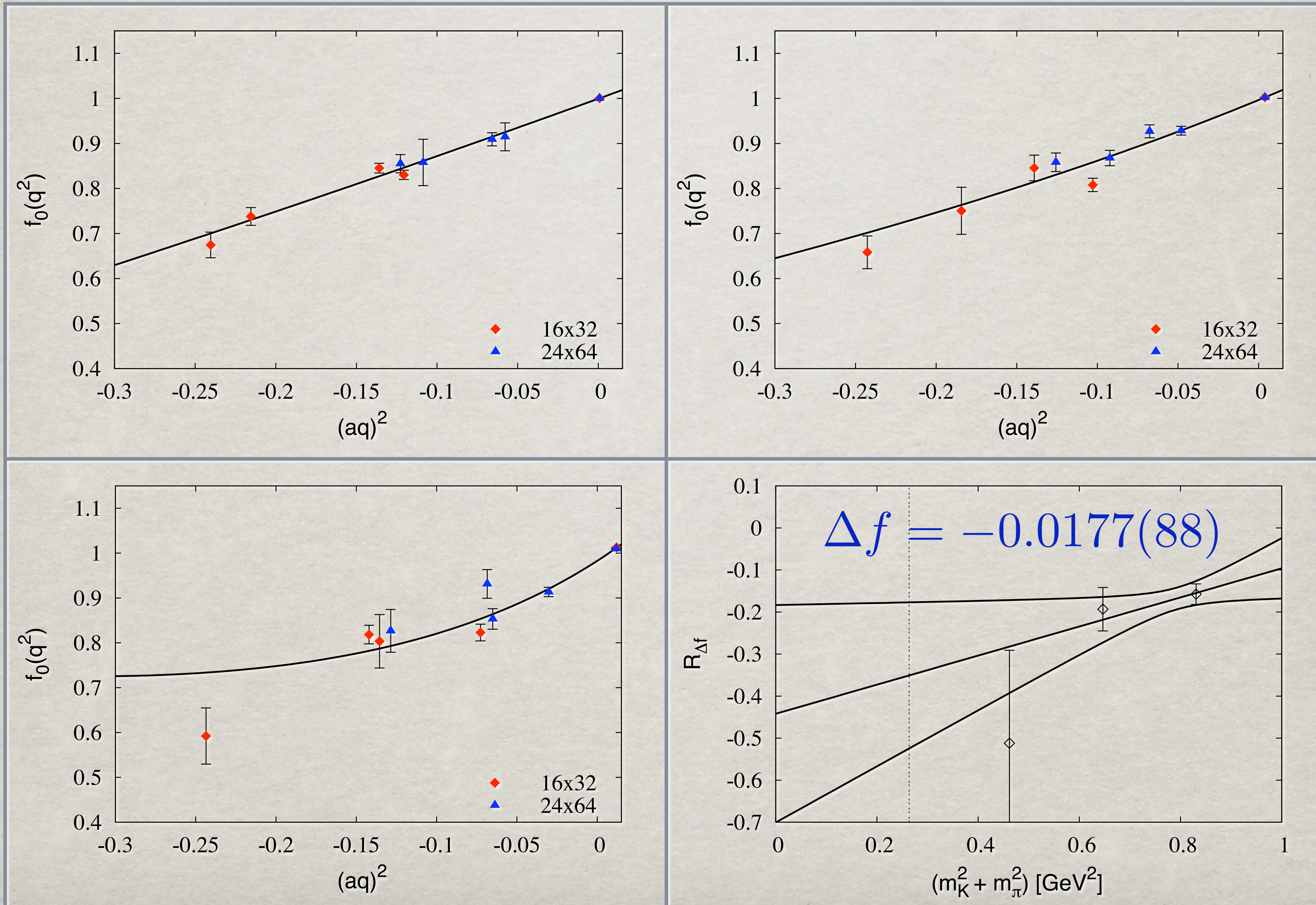
1. Linear: $f_0(q^2) = f_0(0) + a_1 q^2$



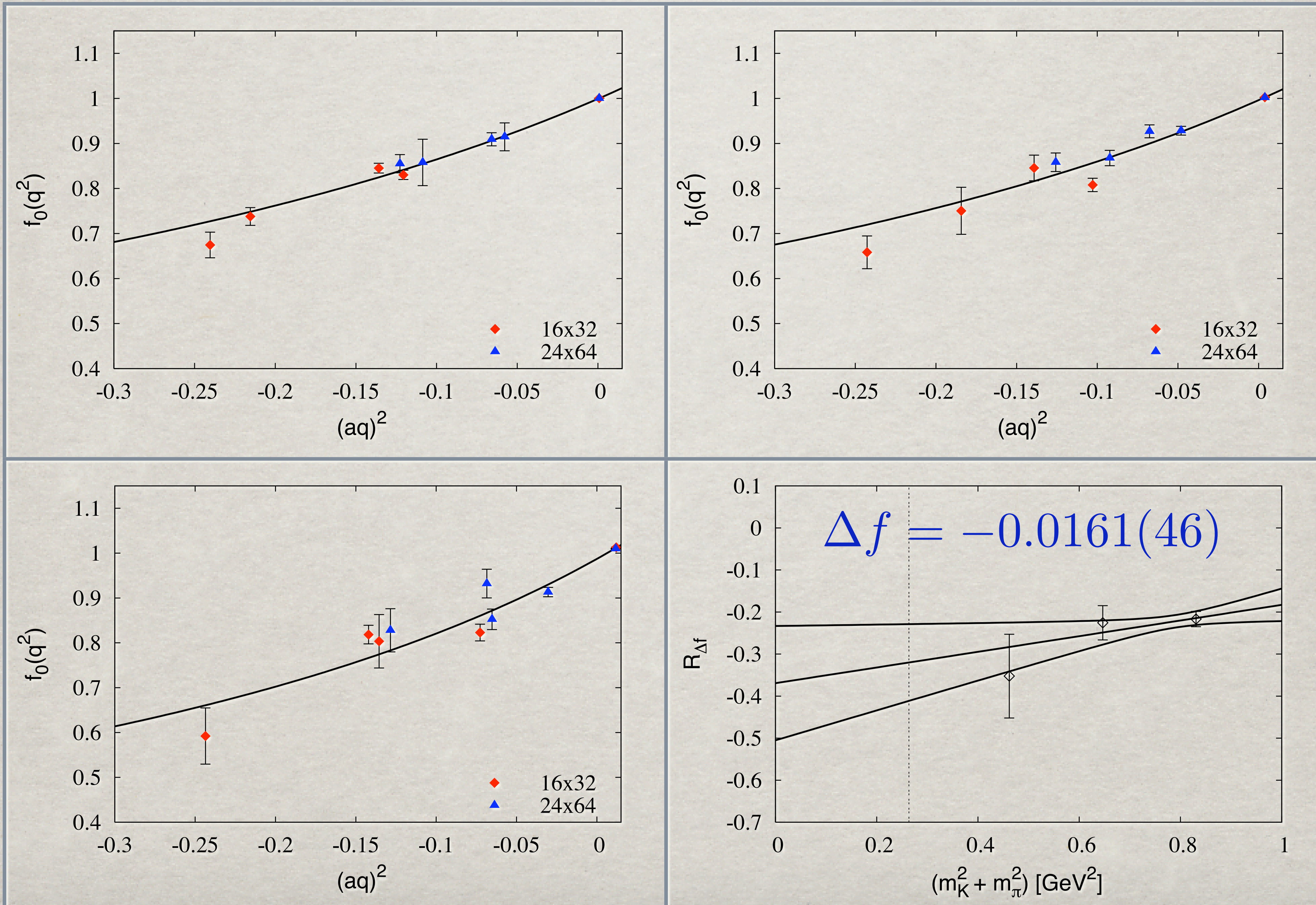
2. Quadratic: $f_0(q^2) = f_0(0) + a_1 q^2 + a_2 q^4$



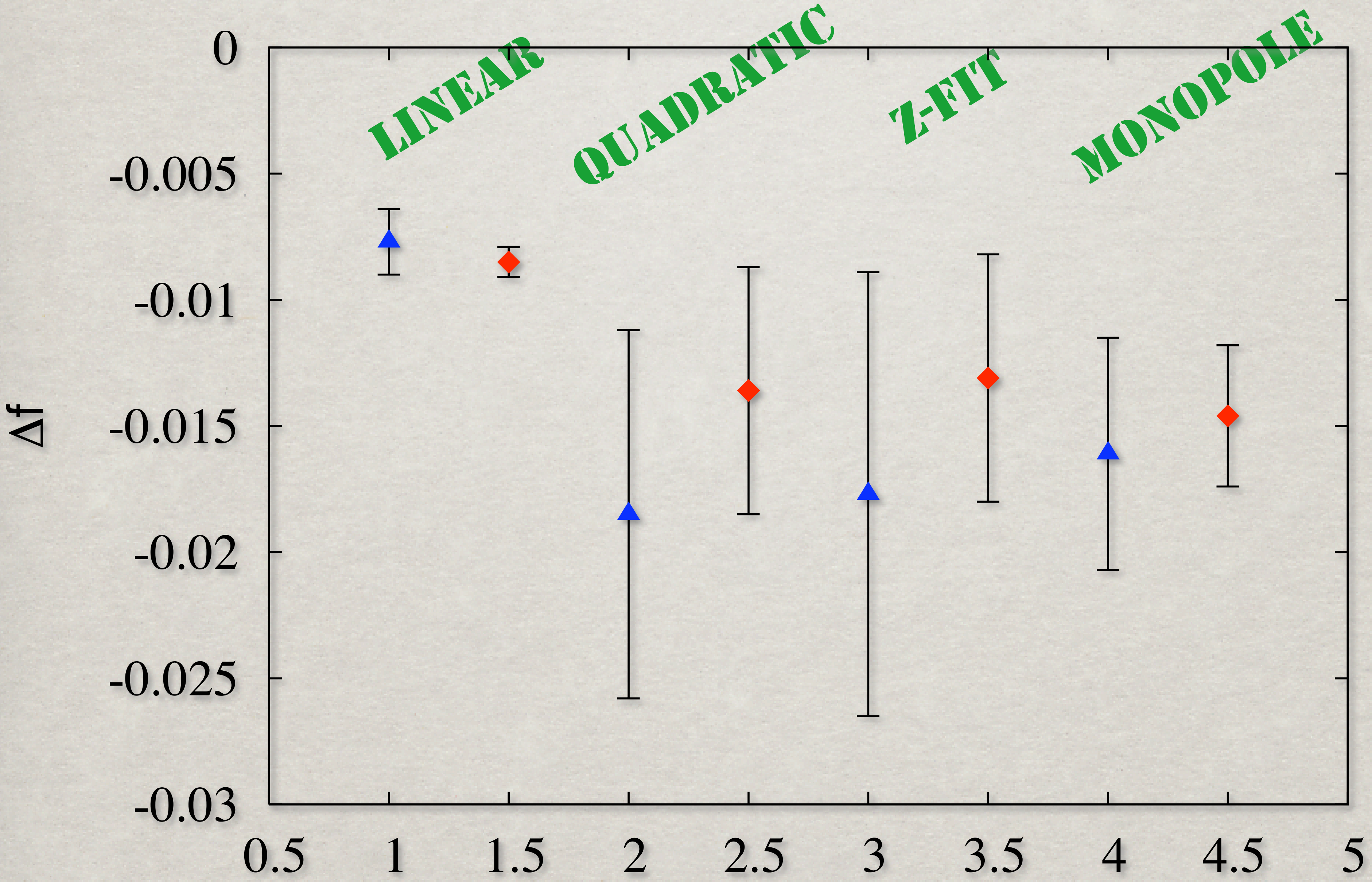
3. Z Fit: $f_0(t) = \frac{1}{\phi(t,t_0,Q^2)} \sum_{k=0}^{\infty} a_k(t_0, Q^2) z(t, t_0)^k$



4. Pole: $f_0(q^2) = f_0(0)/(1 - q^2/M^2)$



Comparison



LINEAR
QUADRATIC
Z-FIT
MONOPOLE
Δf
0
-0.005
-0.01
-0.015
-0.02
-0.025
-0.03
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

$|V_{us}|$

CKM(2006), KAON(2007)

$$\Delta f = -0.0161(46)(15)(16) \Rightarrow f_+^{K\pi}(0) = 0.9609(51)$$

Using $|V_{us} f_+(0)| = 0.2169(9)$ from experimental decay rate:

$$|V_{us}| = 0.2257(9)_{\rm exp}(12)_{f_+(0)}$$

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - \delta, \quad \delta = 0.00076(62)$$

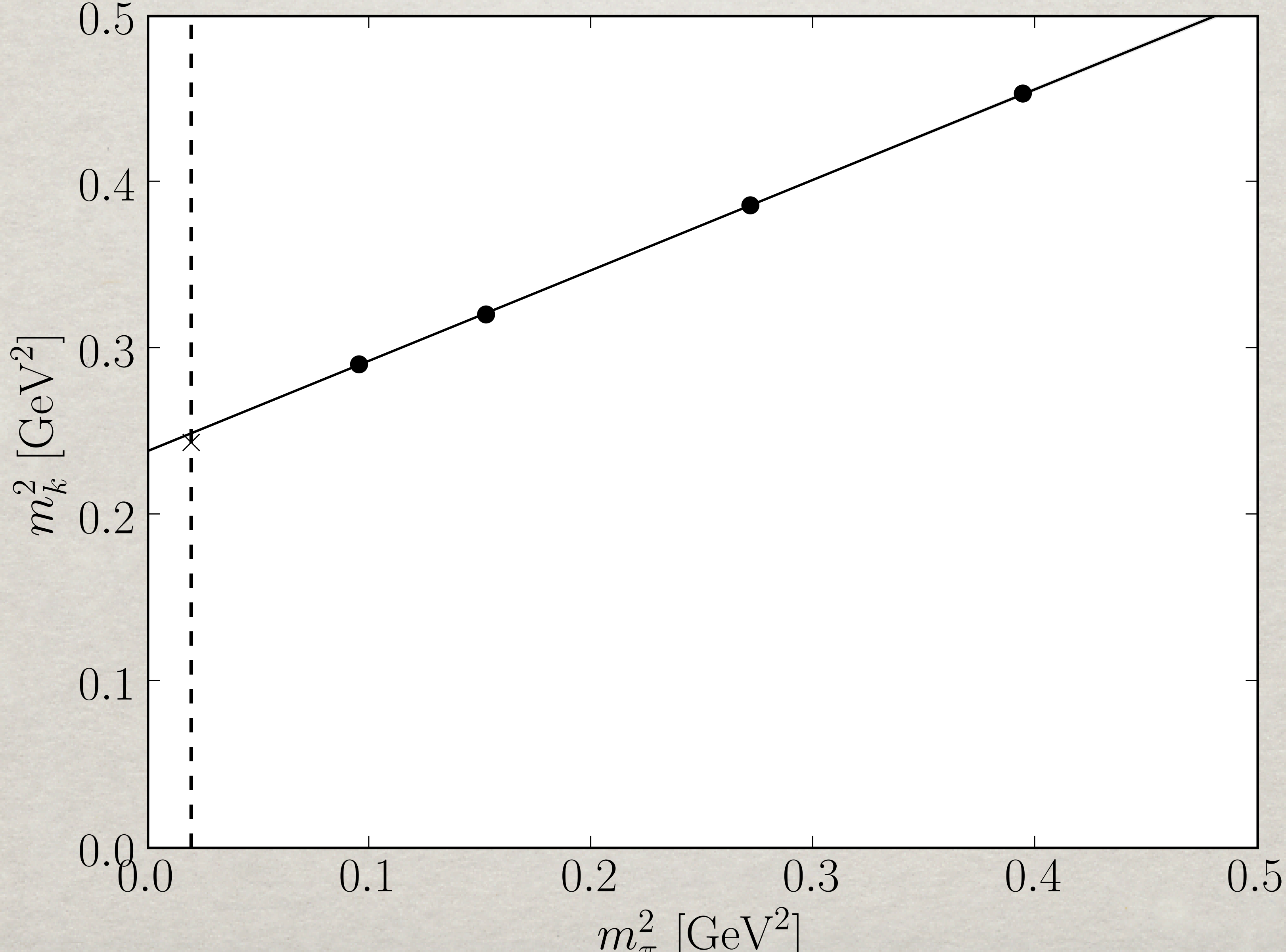
PDG(2006)/LR:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - \delta, \quad \delta = 0.0008(10)$$

New Developments

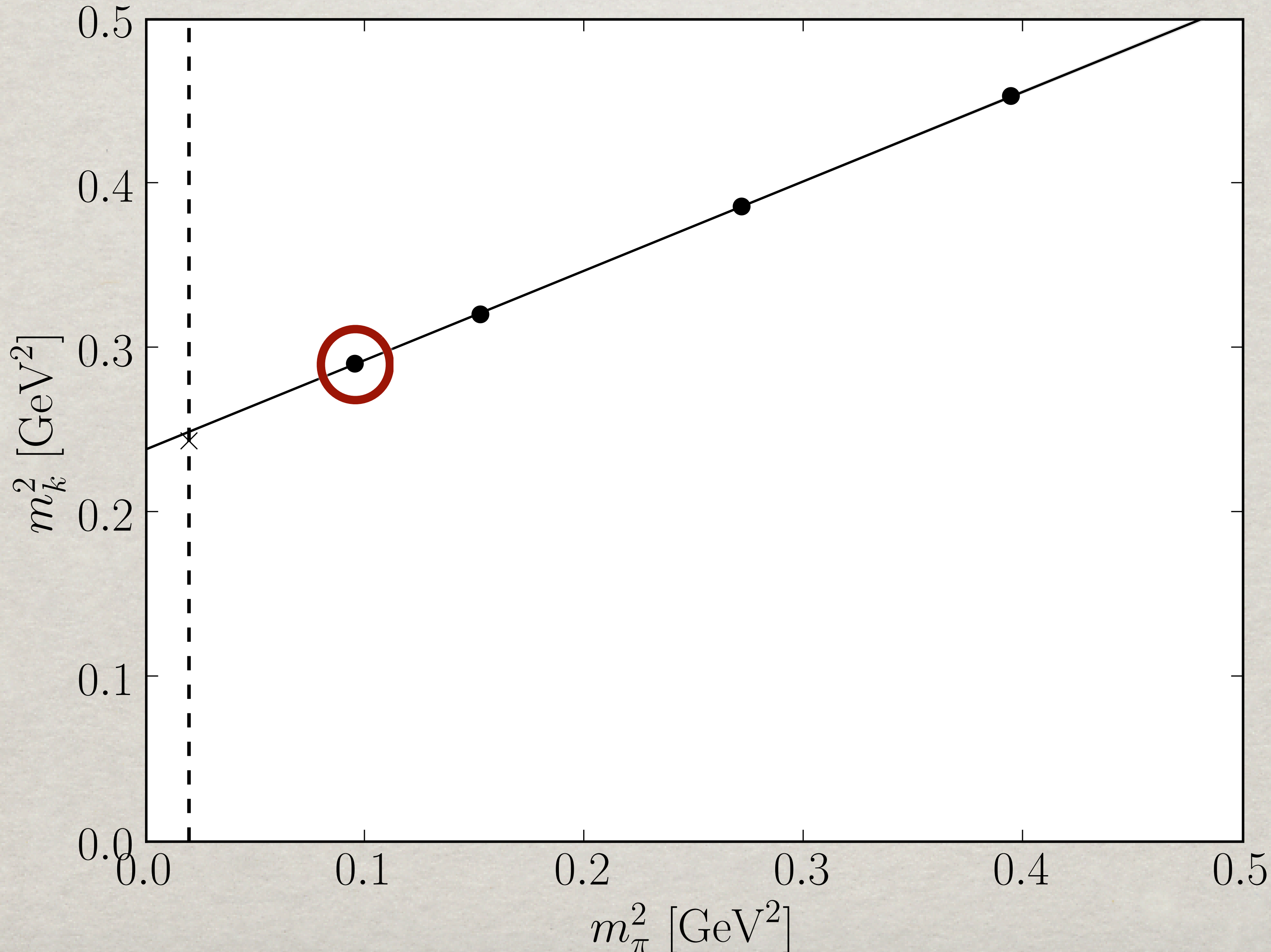
Light Quark Masses

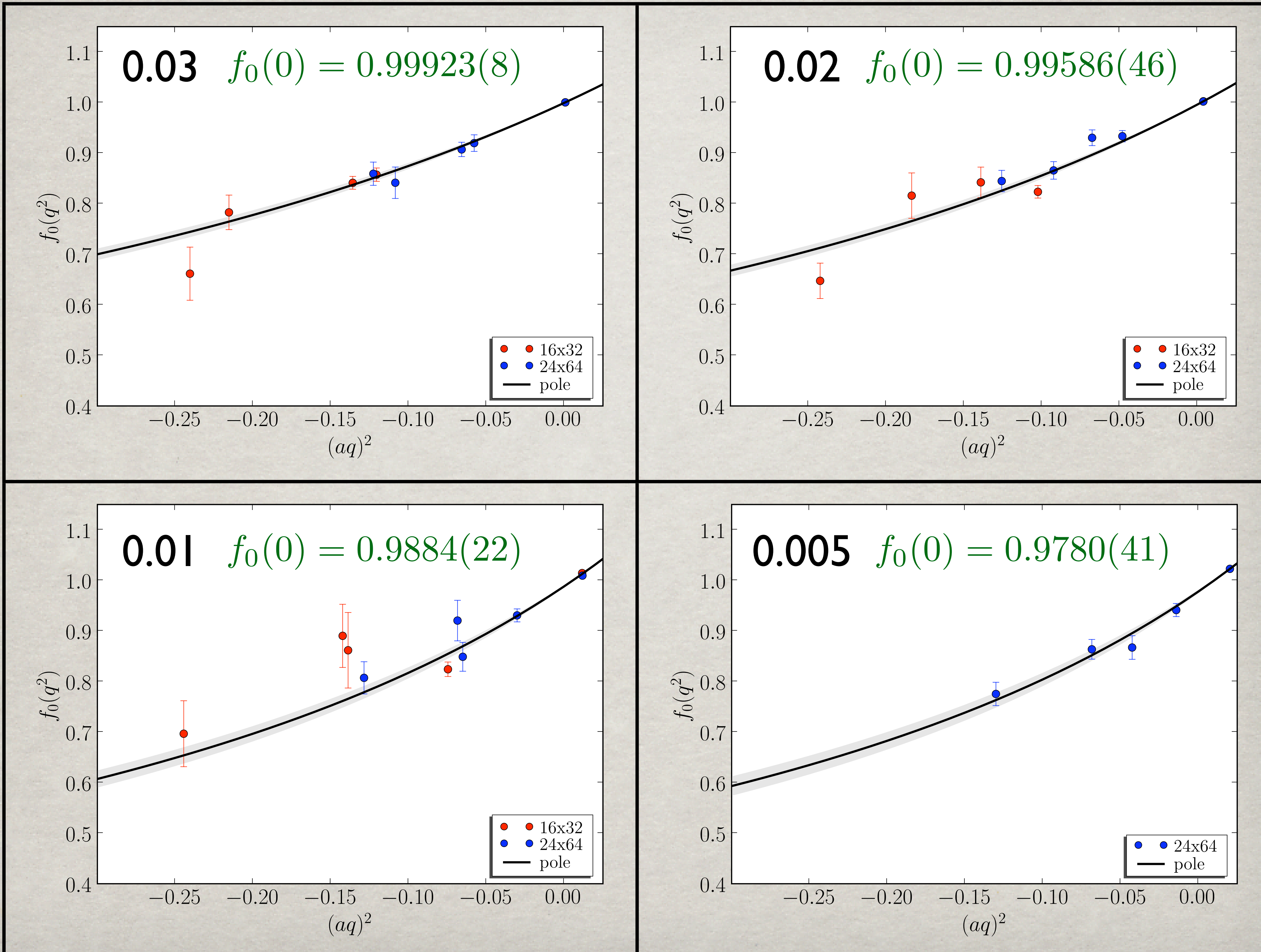
New ensemble at $m_{ud} = 0.005$ $24^3 \times 48$



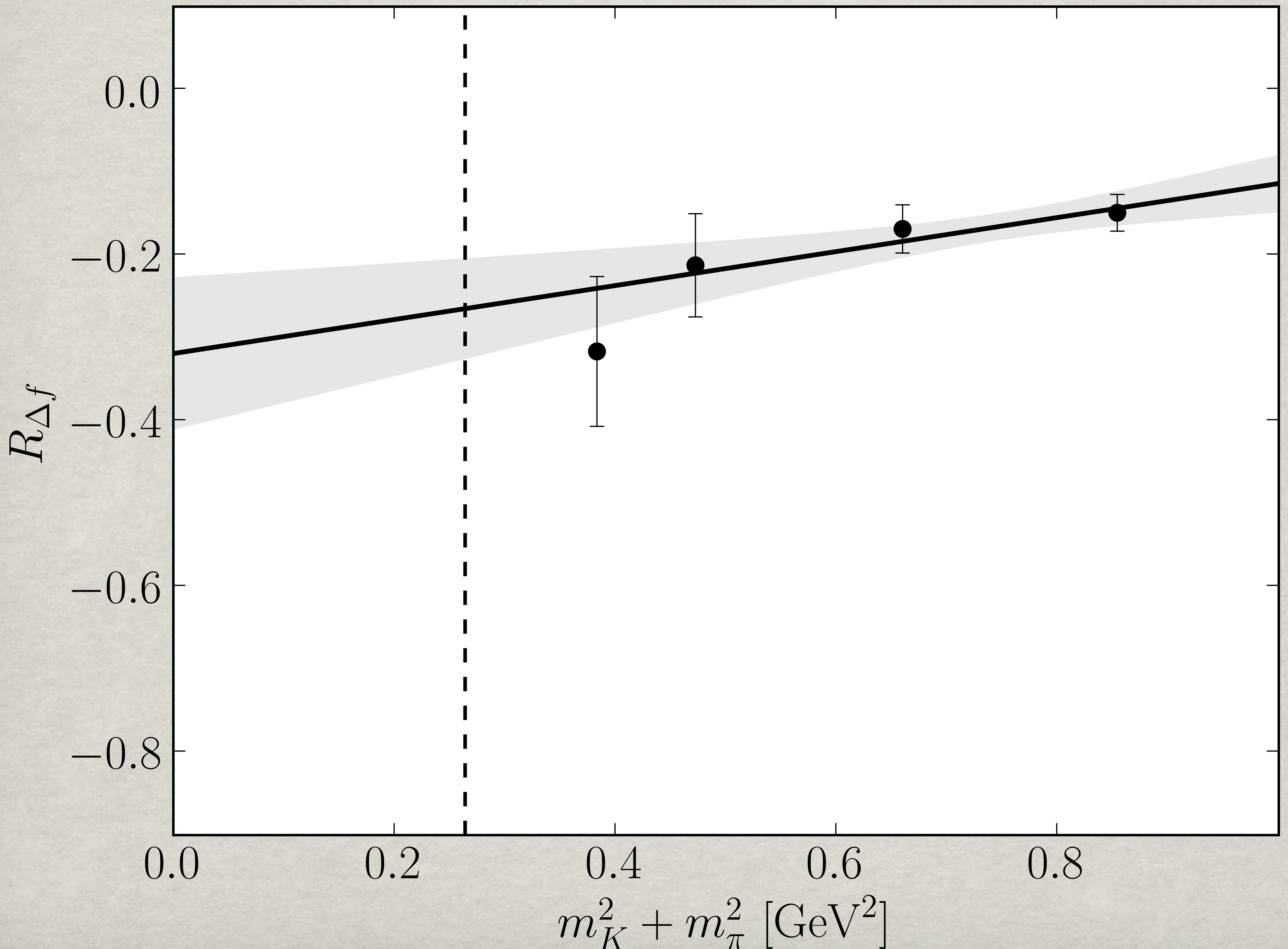
Light Quark Masses

New ensemble at $m_{ud} = 0.005$ $24^3 \times 48$





0.03 $f_0(0) = 0.99923(8)$
0.02 $f_0(0) = 0.99586(46)$
0.01 $f_0(0) = 0.9884(22)$
0.005 $f_0(0) = 0.9780(41)$
$f_0(q^2)$
$(aq)^2$
16x32
24x64
pole


0.0
−0.2
−0.4
−0.6
−0.8
$R_{\Delta f}$
0.0
0.2
0.4
0.6
0.8
$m_K^2 + m_\pi^2$ [GeV^2]

Global fit to K_{l3} data

Usually we proceed by first fitting the form factor with a (eg. pole) ansatz at each simulated quark mass

$$f_0(q^2) = \frac{f_0(0)}{1 - q^2/M^2}$$

Then extrapolate the results to the physical masses

$$f_+(0) = 1 + f_2 + \Delta f$$

$$R_{\Delta f} = \frac{\Delta f}{(M_K^2 - M_\pi^2)^2} = a + b(M_K^2 + M_\pi^2)$$

In an attempt to get as much information out of the lattice data as possible, we attempt to fit the q^2 and quark mass dependencies simultaneously

$$\frac{1+f_2+(m_K^2-m_\pi^2)^2(A_1+A_2(m_K^2+m_\pi^2))}{1-\frac{q^2}{M_0+M_1(m_K^2+m_\pi^2)}}$$

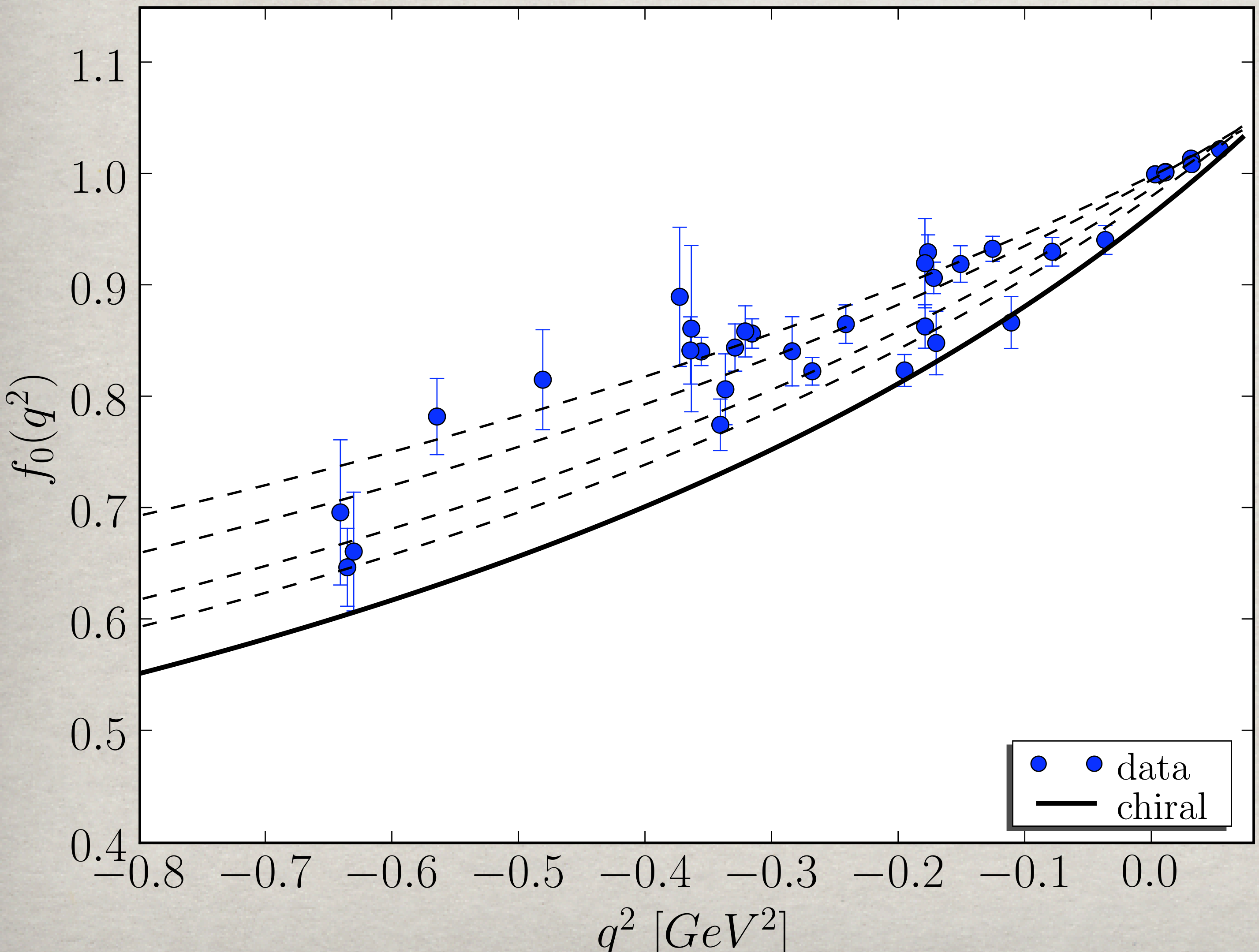
where A_0, A_1, M_0 and M_1 are fit parameters and

$$f_2=\frac{3}{2}H_{\pi K}+\frac{3}{2}H_{\eta K}$$

$$H_{PQ}=-\frac{1}{64\pi^2 f_\pi^2}\left[M_P^2+M_Q^2+\frac{2M_P^2M_Q^2}{M_P^2-M_Q^2}\log\left(\frac{M_Q^2}{M_P^2}\right)\right]$$

The q^2 dependence

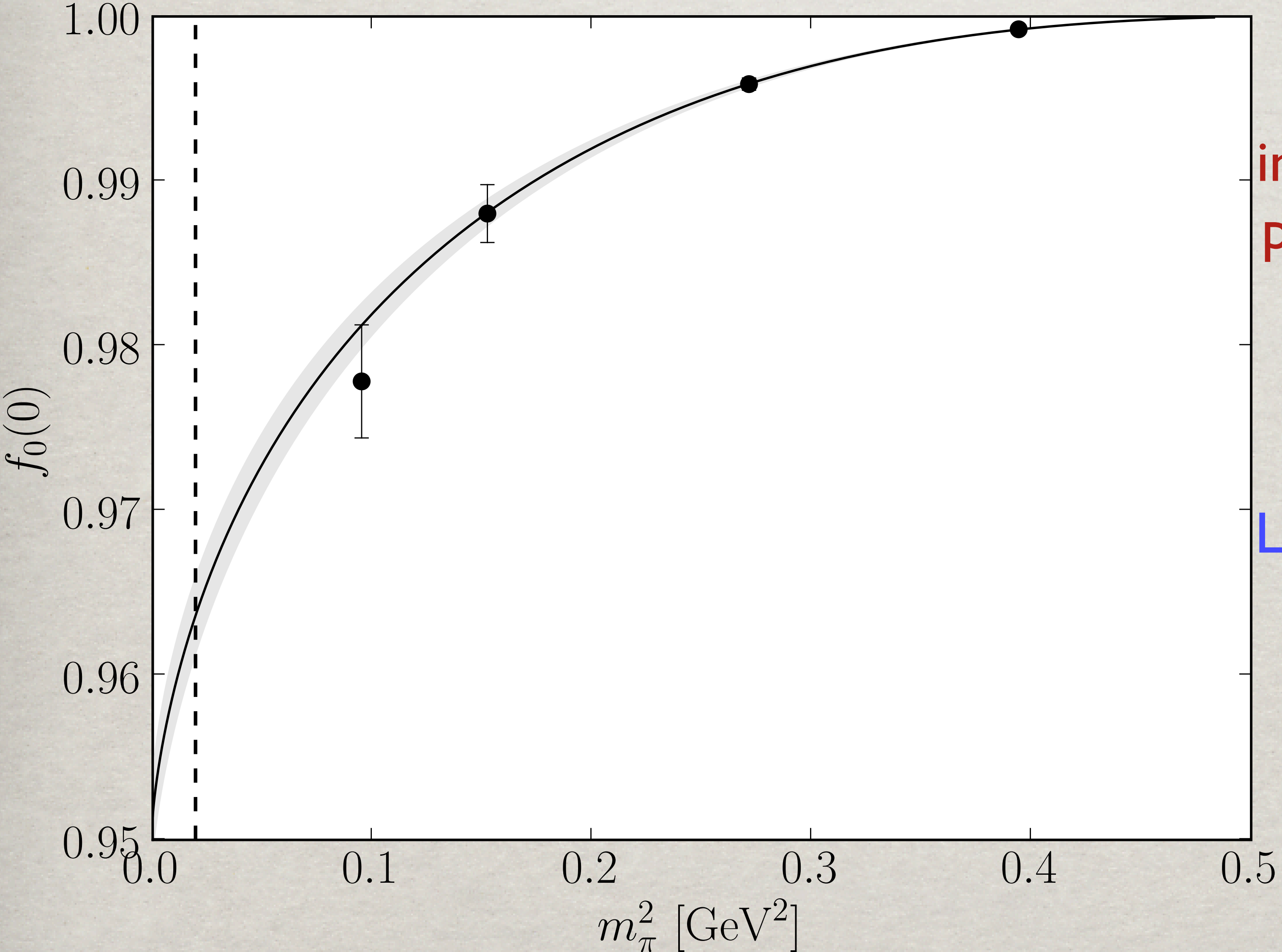
The data points are the lattice results from all 4 masses



Dotted lines are constructed using the fitted parameters inserted into the previous ansatz and evaluated at each of the 4 quark masses

Solid line: as above but for the physical meson masses

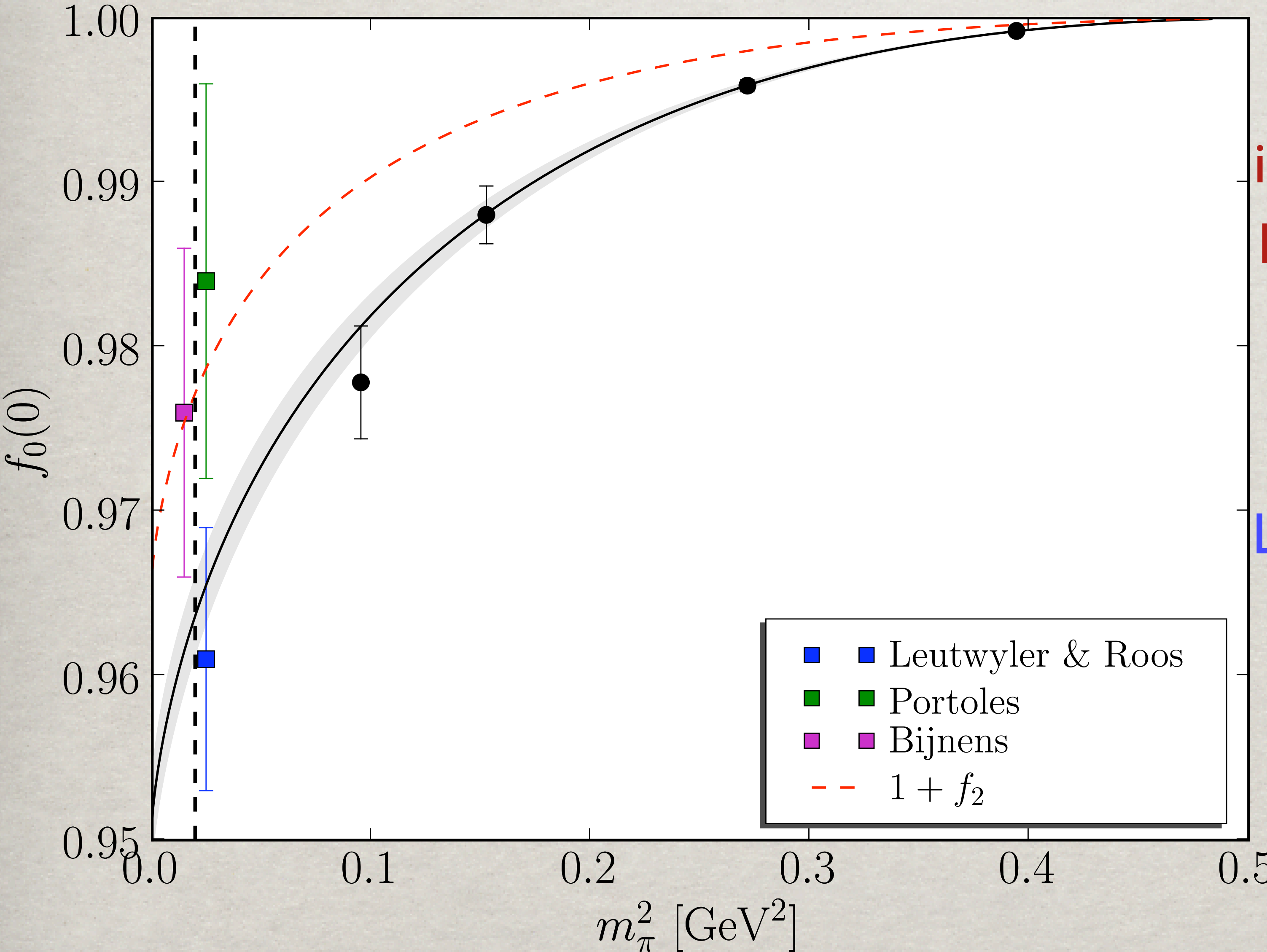
The quark mass dependence



Points come from individual (non-global) pole fits to the lattice data at each quark mass

Line is the result from the global ansatz, evaluated at $q^2=0$

The quark mass dependence



Points come from individual (non-global) pole fits to the lattice data at each quark mass

Line is the result from the global ansatz, evaluated at q^2=0

Summary and Future Work

Preliminary $N_f = 2+1$ result for

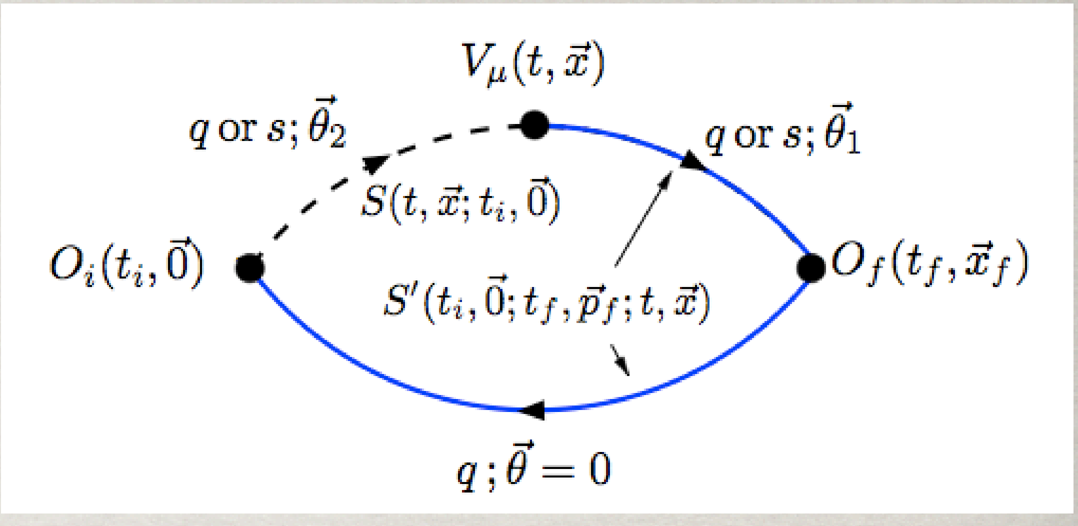
- agrees well with L/R result
- no obvious finite size effects
- small statistical error
- progress towards controlling systematics

Further Improvements

- Lighter quark masses $(am_q = 0.005)$
- Another $\beta \rightarrow$ continuum limit
- Twisted boundary conditions $\longrightarrow$ smaller q^2

Twisted Boundary Conditions

hep-lat/0703005



- On a periodic lattice with spatial volume L^3, momenta are discretised in units of $2\pi/L$
- Modify boundary conditions on the valence quarks

$$\psi(x_k+L)=e^{i\theta_k}\psi(x_k),\qquad (k=1,2,3)$$

- allows to tune the momenta continuously $\vec{p}_{\rm FT}+\vec{\theta}/L$
- to obtain $q^2=0$

$$q^2=(p_f-p_i)^2=\left\{[E_f(\vec{p}_f)-E_i(\vec{p}_i)]^2-\left[(\vec{p}_{{\rm FT},f}+\vec{\theta}_f/L)-(\vec{p}_{{\rm FT},i}+\vec{\theta}_i/L)\right]^2\right\}$$

Twisted Boundary Conditions

$f_0^{K\pi}(0)$	0.02	0.01
old	0.9951(24)	0.9827(29)
new	0.9926(34)	0.9884(34)