Non-abelian vortices in $\mathcal{N} = 2$ gauge theories

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In collaboration with
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   - Simplest example: $SU(2) \times U(1)$
   - Vortex solutions in $U(N)$
2. Non-abelian vortices in $SO(N) \times U(1)$
   - Construction of vortex solutions
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Simplest example: $SU(2) \times U(1)$ theory

- $\mathcal{N} = 2$ theory with gauge group $SU(2) \times U(1)$
- Fayet-Iliopoulos term ξ
- N_f massless hypermultiplets $q_A, \tilde{q}_A$ in $(\underline{2}, 1), (\underline{\bar{2}}, -1)$ reps
- $U(N_f)$ flavor symmetry

Bosonic part of the Lagrangian (neglecting ϕ^0, ϕ^b):

$$\mathcal{L}_{bos} = -\frac{1}{4g_1^2} F^{0\mu\nu} F^0_{\mu\nu} - \frac{1}{4g_2^2} F^{b\mu\nu} F^b_{\mu\nu} + \mathcal{D}_\mu q_A^\dagger \mathcal{D}^\mu q^A + \mathcal{D}_\mu \tilde{q}_A \mathcal{D}^\mu \tilde{q}^{A\dagger} +$$

$$-\frac{g_2^2}{8} \left| q_A^\dagger t^b q^A - \tilde{q}_A t^b \tilde{q}^{A\dagger} \right|^2 - \frac{g_1^2}{24} \left| q_A^\dagger q^A - \tilde{q}_A \tilde{q}^{A\dagger} \right|^2 - \frac{g_2^2}{2} \left| \tilde{q}_A t^b q^A \right|^2 - \frac{g_1^2}{6} \left| \tilde{q}_A q^A - \xi \right|^2$$

Higgs phase: color-flavor locked vacuum $q_i{}^A = \tilde{q}_i{}^{A\dagger} = \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Form of the vacuum invariant under $SU(2)_{C+F}$ global symmetry

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Non-abelian vortex equations

Look for solutions not depending on z, t. Ansatz $q = \tilde{q}^\dagger$.

Bogomolny bound for the tension $T \geq \left| \int d^2x \frac{\xi}{2\sqrt{3}} \varepsilon_{ij} F^0_{ij} \right|$

Non-abelian BPS equations: $\begin{cases} F^b_{ij} + \frac{g_2^2}{2} \varepsilon_{ij} q^\dagger_A t^b q^A = 0 \\ F^0_{ij} + \frac{g_1^2}{\sqrt{3}} \varepsilon_{ij} (q^\dagger_A q^A - \xi) = 0 \\ \mathcal{D}_i q^A + i \varepsilon_{ij} \mathcal{D}_j q^A = 0 \end{cases}$

Ansatz for the vortex solution

$$\begin{cases} A^a_i = -h_a(r) \varepsilon_{ij} \frac{r_j}{r^2} \\ q = \begin{pmatrix} e^{in_1\vartheta} \varphi_1(r) & 0 \\ 0 & e^{in_2\vartheta} \varphi_2(r) \end{pmatrix} \end{cases}$$

$$h_1(r) = h_2(r) = 0$$
$$h_3(0) = 0, \; h_3(\infty) = n_1 - n_2$$
$$h_0(0) = 0, \; h_0(\infty) = \sqrt{3}(n_1 + n_2)$$
$$\varphi_1(\infty) = \varphi_2(\infty) = \sqrt{\frac{\xi}{2}}$$

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Non-abelian vortex solutions

- Solutions are classified by positive integers (n_1, n_2). Their tension is
$$T = 2\pi\xi \left|n_1 + n_2\right|$$
- Topological classification $\pi_1\left(\frac{SU(2)\times U(1)}{\mathbb{Z}_2}\right) = \pi_1(U(2)) = \mathbb{Z}$
Minimal loops: $n_1 + n_2 = 1$, half winding in $U(1)$ and half in $SU(2)$
- From every solution we can build other solutions by applying $SU(2)_{C+F}$ transformations $q'(r,\vartheta) = U_{C+F}\, q(r,\vartheta) U^\dagger_{C+F}$
- $SU(2)_{C+F}$ transformations interpolate between fundamental vortices $(1,0)$ and $(0,1)$ $\rightarrow$ solutions in $SU(2)/U(1) \cong \mathbb{C}P^1$
- Moduli space of fundamental vortices ($T = 2\pi\xi$) is $\mathbb{C} \times \mathbb{C}P^1$ (position $\times$ internal d.o.f.) with $SU(2)$ isometry

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Non-abelian vortices in $U(N)$

- Ansatz $q = \begin{pmatrix} e^{in_1\vartheta}\varphi_1(r) & 0 & 0 & \cdots \\ 0 & e^{in_2\vartheta}\varphi_2(r) & 0 & \cdots \\ 0 & 0 & e^{in_3\vartheta}\varphi_3(r) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$
- Solutions classified by positive integers $(n_1, n_2, n_3 \dots)$ with tension $T = 2\pi\xi\,|n_1 + n_2 + n_3 + \dots|$
- Topological classification $\pi_1\left(\frac{SU(N)\times U(1)}{\mathbb{Z}_N}\right) = \pi_1(U(N)) = \mathbb{Z}$
  Minimal loops: $n_1 + n_2 + n_3 + \dots = 1$, $1/N$ winding in $U(1)$ and $1/N$ in $SU(N)$
- $SU(N)_{C+F}$ transformations on vortices
- Fundamental vortices: $(1, 0, 0 \dots)$ and its $SU(N)_{C+F}$ orbit
  Moduli space $\mathbb{C} \times \mathbb{C}P^{N-1}$ with $SU(N)$ isometry (Fubini-Study metric)

Moduli spaces of $U(N)$ non-abelian vortices

Moduli matrix approach

BPS equations for vortices:

$$(\mathcal{D}_1 + i\mathcal{D}_2)\, q = 0, \quad F_{12}^{(0)} + \frac{e^2}{2}\,(c\,\mathbf{1}_N - q\,q^\dagger) = 0, \quad F_{12}^{(a)} + \frac{g_N^2}{2}\, q_i^\dagger\, t^a\, q_i = 0$$

These equations can be solved as

$$q = S^{-1}(z,\bar{z})\, H_0(z), \quad A_1 + i\,A_2 = -2\,i\,S^{-1}(z,\bar{z})\,\bar{\partial}_z S(z,\bar{z})$$

where S is an $N \times N$ invertible matrix over the whole z plane, and H_0 (moduli matrix) is holomorphic in z, defined modulo a nonsingular holomorphic $N \times N$ matrix $V(z)$:

$H_0(z) \to V(z)\, H_0(z), \quad S(z,z^*) \to V(z)\, S(z,z^*)$

One more equation for $\Omega = S\,S^\dagger$, but expected to give no additional moduli: $\partial_z\,(\Omega^{-1}\partial_{\bar{z}}\,\Omega) = -\frac{g_N^2}{2}\,\mathrm{Tr}\,(\,t^a\,\Omega^{-1}\,q\,q^\dagger)\,t^a - \frac{e^2}{4N}\,\mathrm{Tr}\,(\,\Omega^{-1}q\,q^\dagger - \mathbf{1})$

Moduli space for composites of k minimal vortices ($T = 2\pi\xi k$):

$\{H_0(z) |\, \det(H_0) \sim z^k,\ z \to \infty\} \,/\, \{V(z) |\, \det(V) = const \neq 0\}$

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$U(N)$ vortices of higher winding

Physically $\det(H_0) \sim (z - z_1)(z - z_2)(z - z_3) \ldots$ describes position of vortices

Consider the case of coincident vortices:

$U(2)$ example

In $k = 2$ case, moduli space has three patches

$$H_0 \simeq \begin{pmatrix} z^2 & 0 \\ -a'z - b' & 1 \end{pmatrix}; \quad \begin{pmatrix} z - \phi & \eta \\ \bar{\eta} & z + \phi \end{pmatrix}; \quad \begin{pmatrix} 1 & -az - b \\ 0 & z^2 \end{pmatrix}$$

with constraint $\phi^2 + \eta\bar{\eta} = 0$ (Z_2 singularity at the origin)

These patches cover the manifold $W\mathbb{C}P^2_{(2,1,1)}$

In $U(N)$ theories, moduli space for $k = 2$ coincident vortices is $WGr^{(1,\ldots 1,0)}_{N+1,2}$ where $Gr_{N+1,2} \simeq \frac{SU(N+1)}{SU(N-1)\times SU(2)\times U(1)}$

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Developments in the study of non-abelian vortices

- Works in $\mathcal{N} = 2$ $SU(N)$ theories:
  - BPS local and semilocal vortex solutions
  - Complete moduli spaces obtained with different tecniques (but unknown metric for higher winding)
  - Correspondence between BPS states in 2d and 4d field theories:

    Kinks on the worldsheet $\leftrightarrow$ Vortex junctions $\leftrightarrow$ Monopoles in the 4d theory
  - Reconnection of cosmic strings
  - *Monopole confinement*
- Recent directions:
  - *Non-abelian vortices in $SO(N)$ theories*
  - Non-abelian vortices in $\mathcal{N} = 1$ SQCD and non-SUSY theories

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Non-abelian vortices in $SO(N) \times U(1)$

$\mathcal{N} = 2$ theory with gauge group $SO(2N) \times U(1)$ and fields $q_A, \tilde{q}_A^\dagger$ in the $(\underline{2N}, +1)$ representation

$$
\begin{aligned}
\mathcal{L} \;=\;& -\frac{1}{4g_1^2} F^{0\mu\nu} F^0_{\mu\nu} - \frac{1}{4g_{2N}^2} F^{b\mu\nu} F^b_{\mu\nu} + |\mathcal{D}_\mu q_A|^2 + \left|\mathcal{D}_\mu \tilde{q}_A^\dagger\right|^2 \\
& - \frac{g_{2N}^2}{2} \left| q_A^\dagger t^b q_A - \tilde{q}_A t^b \tilde{q}_A^\dagger \right|^2 - 2 g_{2N}^2 \left| \tilde{q}_A t^b q_A \right|^2 \\
& - \frac{g_1^2}{4} \left| q_A^\dagger q_A - \tilde{q}_A \tilde{q}_A^\dagger \right|^2 - g_1^2 \left| \tilde{q}_A q_A - \xi \right|^2 + \cdots
\end{aligned}
$$

$$
\langle q \rangle = \langle \tilde{q}^\dagger \rangle = \sqrt{\frac{\xi}{2N}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ i & -i & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & i & -i & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

Vacuum invariant under $SO(2N)_{C+F}$

Ansatz for the solutions

$$A_i = h_a(r)\, t^a\, \varepsilon_{ij} \frac{r_j}{r^2}$$

t^a generators of $SO(2N)$ Cartan subalgebra

$$q(r,\vartheta) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{in_1^+\vartheta}\varphi_1^+(r) & e^{in_1^-\vartheta}\varphi_1^-(r) & 0 & 0 & \cdots \\ ie^{in_1^+\vartheta}\varphi_1^+(r) & -ie^{in_1^-\vartheta}\varphi_1^-(r) & 0 & 0 & \cdots \\ 0 & 0 & e^{in_2^+\vartheta}\varphi_2^+(r) & e^{in_2^-\vartheta}\varphi_2^-(r) & \cdots \\ 0 & 0 & ie^{in_2^+\vartheta}\varphi_2^+(r) & -ie^{in_2^-\vartheta}\varphi_2^-(r) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Finite-energy conditions: $\varphi_a^\pm(\infty) = \sqrt{\frac{\xi}{2N}}$

$$n_a^\pm = n^{(0)} \mp n^{(a)}, \qquad n^{(0)} \equiv \frac{1}{\sqrt{2}} h_0(\infty); \quad n^{(a)} \equiv \frac{1}{\sqrt{2}} h_a(\infty)$$

$$\Rightarrow N_0 = n_a^+ + n_a^-, \qquad T = 2\pi\xi |N_0|$$

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Moduli space of $SO(N)$ non-abelian vortices

Vortex solutions are classified by $2N + 1$ integers N_0,$n_a^{\pm}$ which satisfy the following conditions:

$$n_a^+ + n_a^- = N_0 \ , \quad \forall a$$
$$\text{sign}(n_a^+) = \text{sign}(n_a^-) = \text{sign}(N_0) \ , \quad \forall a$$

Fundamental vortices belong to two classes of 2^{N-1} elements:

$$N_0 = 1, \qquad \begin{pmatrix} n_1^+ & n_1^- \\ n_2^+ & n_2^- \\ \vdots & \vdots \\ n_{N-1}^+ & n_{N-1}^- \\ n_N^+ & n_N^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{pmatrix} , \ldots ,$$

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Vacuum invariant under $SO(2N)_{C+F}$ symmetry; the two classes above belong to two different orbits.
Each solution is invariant under a subgroup $U(N) \subset SO(2N)_{C+F}$
$\Rightarrow$ moduli space composed by a pair of coset spaces
$\mathcal{M} = SO(2N)/U(N)$

Moduli space for higher windings not known

Topology of $SO(N)$ non-abelian vortices

The gauge group is $\frac{SO(2N)\times U(1)}{Z_2}$ which has a nontrivial homotopy group

$$\pi_1\left(\frac{SO(2N)\times U(1)}{\mathbb{Z}_2}\right) = \mathbb{Z}\times\mathbb{Z}_2$$

This is because of the equivalence relation $(\mathbf{1},-1)\simeq(-\mathbf{1},1)$
$\Rightarrow$ the minimal nontrivial element of π_1 corresponds to half winding in $U(1)$ and half winding in $SO(2N)$. But there are *two* inequivalent possibilities for winding in $SO(2N)$, so there are two minimal elements.

The two classes of minimal vortices correspond to these two topologically inequivalent paths $\Rightarrow$ no solutions interpolating between these classes.

Non-abelian vortices in $SO(N+1)$

Similar ansatz $q(r,\vartheta) = \begin{pmatrix} \mathbf{M}_1(r,\vartheta) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & \mathbf{M}_N(r,\vartheta) & 0 \\ 0 & \cdots & 0 & e^{i\hat{n}\vartheta}\hat{\varphi}(r) \end{pmatrix}$

Finite energy gives the condition $\hat{n} = \frac{h_0(\infty)}{\sqrt{2}} = \frac{N_0}{2}$ that implies N_0 must be even ($n^{(0)}$ integer)

Consistent with topology because $\pi_1(SO(2N+1) \times U(1)) = \mathbb{Z} \times Z_2$ where the minimum loop is a complete winding around $U(1)$

$SO(2N+1)_{C+F}$ symmetry, all minimal vortices belong to the same orbit $\Rightarrow$ moduli space $SO(2N+1)/U(N)$

Non-abelian monopoles

Non-abelian monopoles are generalizations of t'Hooft-Polyakov monopoles from the breaking $SU(2) \to U(1)$ to the case of theories with gauge symmetry breaking pattern $G \longrightarrow H$ with H non-abelian.

Can be seen as t'Hooft-Polyakov monopoles of some $SU(2)$ subgroup embedded in G which gets broken to $U(1) \subset H$.

Example: $SU(3) \to SU(2) \times U(1)$

$$\langle \phi \rangle \sim \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}$$

Monopoles embedded in broken $SU(2)$ subgroups
$\mathbb{C}P^1$ moduli space with $SU(2)$ isometry

Problems with non-abelian: non-normalizable zero-modes, topological obstructions in defining global electric charge

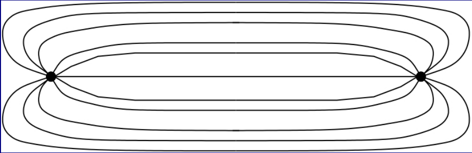
Monopole confinement

Consider a theory with this pattern of symmetry breaking:

$$G \xrightarrow{v_1} H \xrightarrow{v_2} 1 \qquad v_1 \gg v_2$$

- Existence of stable monopoles depend on the group $\pi_2(G/H)$, so guaranteed only in the limit $v_2 \to 0$
- Existence of stable vortices depend on the group $\pi_1(H)$ and only guaranteed if $v_2 \neq 0$

What is the fate of monopoles when $v_2 \neq 0$? Monopole-antimonopole pairs become confined by a vortex string (or, in the case of a single monopole, its magnetic flux becomes a vortex at distances greater than $1/v_2$).



Topological correspondence

The topological correspondence

$$\pi_2(G/H) = \pi_1(H)/\pi_1(G)$$

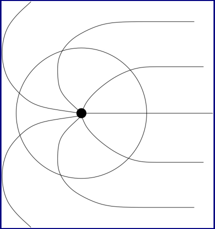
shows that vortices and monopoles are topologically related.

- If the gauge group G is simply connected, there is a one-to-one relation between regular monopoles and vortices
- If the gauge group is *not* simply connected, then for each monopole (regular but also Dirac) there is a corresponding vortex

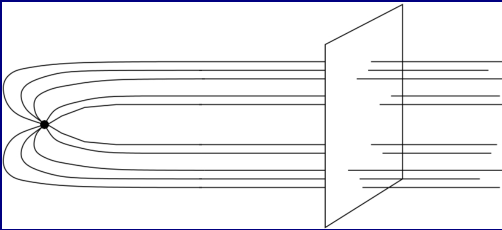
Flux matching

To obtain a precise correspondence between monopoles and vortices, we can match their magnetic fluxes:

Flux of the monopole at scale $1/v_1$



Flux of the vortex at scale $1/v_2$ and far from the monopole



Easy for the abelian flux, not always easy for the non-abelian flux

Monopole-vortex correspondence

Conjecture: there is a strong correspondence between minimal non-abelian monopoles and non-abelian vortices arising as approximately BPS solitons in theories with gauge symmetry breaking pattern $G \longrightarrow H \longrightarrow 1$

Example: $SU(3) \longrightarrow SU(2) \times U(1)$

Both minimal monopoles and minimal vortices are described by $\mathbb{C}P^1$ with $SU(2)$ isometry.

Topology: $\pi_2(SU(3)/U(2)) = \pi_1(U(2)) = \mathbb{Z}$

Abelian and non-abelian magnetic fluxes match correctly.

Note that the isometries have different origin: $SU(2)_C$ transformations on monopoles, global $SU(2)_{C+F}$ transformations on vortices

Monopole-vortex correspondence

Conjecture: there is a strong correspondence between minimal non-abelian monopoles and non-abelian vortices arising as approximately BPS solitons in theories with gauge symmetry breaking pattern $G \longrightarrow H \longrightarrow 1$

Example: $SU(3) \longrightarrow SU(2) \times U(1)$

Both minimal monopoles and minimal vortices are described by $\mathbb{C}P^1$ with $SU(2)$ isometry.

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Explicit examples: $SU(N+1) \to U(N) \to 1$

$\mathcal{N} = 2$ high-energy lagrangian with gauge group $SU(N+1)$

$$\mathcal{L}_{SU(N+1)} = \frac{1}{8\pi} \mathrm{Im}\, S_{cl} \left[\int d^4\theta\, \Phi^\dagger e^V \Phi + \int d^2\theta\, \frac{1}{2} W W \right] + \mathcal{L}^{(\mathrm{q})} + \int d^2\theta\, \mu\, \mathrm{Tr}\, \Phi^2$$

$$\mathcal{L}^{(\mathrm{q})} = \sum_i \left[\int d^4\theta \left\{ Q_i^\dagger e^V Q_i + \tilde{Q}_i e^{-V} \tilde{Q}_i^\dagger \right\} + \int d^2\theta \left\{ \sqrt{2} \tilde{Q}_i \Phi Q^i + m\, \tilde{Q}_i\, Q^i \right\} \right]$$

$$\Phi = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \dots & m & 0 \\ 0 & \dots & 0 & -N\, m \end{pmatrix}$$

Low-energy lagrangian becomes $U(N)$ theory with FI term $\xi = \mu m$
Need $\mu \ll m$ to have a hierarchy $\sqrt{\xi} \simeq v_2 \ll v_1 \simeq m$

Trivial generalization of $SU(3) \to SU(2) \times U(1)$:
Both minimal monopoles and minimal vortices described by $\mathbb{C}P^{N-1}$
Topology: $\pi_2(SU(N+1)/U(N)) = \pi_1(U(N)) = \mathbb{Z}$
Magnetic fluxes match correctly.

$SO(2N) \to U(N) \to 1$

Symmetry breaking $\Phi = \begin{pmatrix} 0 & i\,v & 0 & 0 & \cdots \\ -i\,v & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & i\,v & \cdots \\ 0 & 0 & -i\,v & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

Monopoles embedded in $SO(4) \sim SU(2) \times SU(2)$ subgroups

Flux matching suggests that monopoles correspond to $k = 2$ vortices classified by $(2, 0, 0 \ldots)$. Both monopoles and these vortices transform as $\mathbb{C}P^{N-1}$. Consistent with topological classification $\pi_2(SO(2N)/U(N)) = \pi_1(U(N))/\mathbb{Z}_2$

However, $(2, 0, 0 \ldots)$ vortices are only a submanifold of $k = 2$ moduli space. In the $U(2)$ case, they form $\mathbb{C}P^1 \subset W\mathbb{C}P^2_{(2,1,1)}$. What about the other vortices? Correspondence does not seem to work here...

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$SO(2N+1) \to U(N) \to 1$

Simplest case $SO(5)$: $\Phi = \begin{pmatrix} 0 & i\,v & 0 & 0 & 0 \\ -i\,v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\,v & 0 \\ 0 & 0 & -i\,v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

New minimal monopoles from $SO(3)$ embeddings, but also interpolating solutions, all degenerate in mass! (E.Weinberg)

Moduli space of these monopoles is a manifold $\mathbb{C}^2/Z_2 \bigcup \mathbb{C}P^1$

Moduli space of $k = 2$ vortices is $W\mathbb{C}P^2_{(2,1,1)}$ which has the same topological structure and singularities!

Flux matching only possible for some solutions, but consistent with this picture. The same for topology:
$\pi_2(SO(2N+1)/U(N)) = \pi_1(U(N))/\mathbb{Z}_2$

$SO(2N+2) \rightarrow SO(2N) \times U(1) \rightarrow 1$

High-energy theory with matter in the *adjoint* rep

$$\Phi = \begin{pmatrix} 0 & iv & 0 & 0 & \cdots \\ -iv & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Monopoles embedded in $SO(3)$ subgroups

Low-energy theory contains massless matter multiplets in the *fundamental* representation

As before, flux matching suggests that monopoles correspond to $N_0 = 2$ vortices

$$\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 2 & 1 & 1 & \ldots & 1 \end{pmatrix}$$

$SO(2N+2) \to SO(2N) \times U(1) \to 1$

Consistent with topology, because

$$\pi_2\left(\frac{SO(2N+2)/}{U(1) \times SO(2N)/Z_2}\right) = \frac{\pi_1(U(1) \times SO(2N)/Z_2)}{Z_2}$$

Both vortices and monopoles seem to form a complex quadric surface $SO(2N)/U(1) \times SO(2N-2)$ with $SO(2N)$ isometry
But impossible to compare because of lack of knowledge about true moduli space for vortices of higher winding

The same problem for $SO(2N+3) \to SO(2N+1) \times U(1) \to 1$

Conclusions

- Monopole confinement:
  - Monopole-vortex correspondence seems good
  - More checks in *SO* and *USp*
  - Which confinement? non-BPS corrections...
  - Explicit metric on moduli space?
- Vortices in $SO(N)$:
  - Explicit solutions available
  - Moduli space for $N_0 > 1$?
  - Semilocal vortices, vortex junctions...

(In general, interesting applications of non-abelian vortices in $\mathcal{N} = 1$ SUSY or less ...)