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Non-Perturbative QFT

Integrability in Four-Dimensional Gauge Theory

Physics of many-body systems,

<u>linear</u> : trivial

<u>non-linear</u> : impossible

exceptions are <u>integrable</u>
<u>systems</u>

∞ # of conserved charges

Integrability in 4d gauge theory, $G = SU(N)$

planar limit : $N \to \infty$, $\lambda = g^2 N$ held fixed

<u>QCD</u> Lipatov 1993

parton evolution equations

anomalous dimensions of composite operators

$\mathcal{N}=4$ SUSY Yang-Mills

- conformal field theory

flow to pure Yang-Mills by adding mass terms

- dual to IIB string theory on $AdS_5 \times S^5$ Maldacena

...evidence for complete integrability of planar theory

Minahan + Zarembo

Beisert + Staudacher

Outline

Integrability;

i) Example: $XXX_{1/2}$ spin chain

ii) $\mathcal{N}=4$ SUSY Yang-Mills

Example $XXX_{1/2}$ spin chain

Heisenberg 1926
(1d model of ferromagnetism)

chain of L spin-1/2 variables
with periodic b.c.

$\times \uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow \times$

length: L $\Rightarrow 2^L$ states

of $\downarrow$: M

Heisenberg Hamiltonian,

$$\hat{H} = \sum_{\ell=1}^{L} \left(I_{\ell,\ell+1} - P_{\ell,\ell+1} \right)$$

$$I\,|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle$$
$$P\,|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle$$

Exact solution

Bethe 1931

Integrability

Faddeev + Taktajan 1981

Spectrum $L \to \infty$

M=0 ferromagnetic vacuum

$|0\rangle \equiv \ldots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\ldots$

$E = 0$

M=1 one "impurity"

$|\ell\rangle \equiv \ldots\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\ldots$

ℓ'th site

magnon:

$$|p\rangle = \sum_{\ell} e^{ip\ell} |\ell\rangle$$

momentum eigenstate $\quad \psi_p(\ell) = e^{ip\ell}$

dispersion relation,

$$\underline{\underline{E(p) = 4\sin^2(p/2)}}$$

$M=2$ two magnons

$$\cdots\cdots \uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow \cdots\cdots$$

$$\longrightarrow p_1 \qquad\qquad\qquad p_2 \longleftarrow$$

scattering states,

$$\psi_{p_1,p_2}(\ell_1,\ell_2) = e^{ip_1\ell_1 + ip_2\ell_2} + S(p_2,p_1)\, e^{ip_1\ell_2 + ip_2\ell_1}$$

incident wave

reflected wave

S-matrix,

$$S(p_1,p_2) = \frac{u(p_1) - u(p_2) + i}{u(p_1) - u(p_2) - i}$$

$$u(p) = \frac{1}{2}\cot\left(\frac{p}{2}\right)$$

- scattering is <u>elastic</u> due to 1+1 dimensional kinematics

- also find <u>boundstates</u>

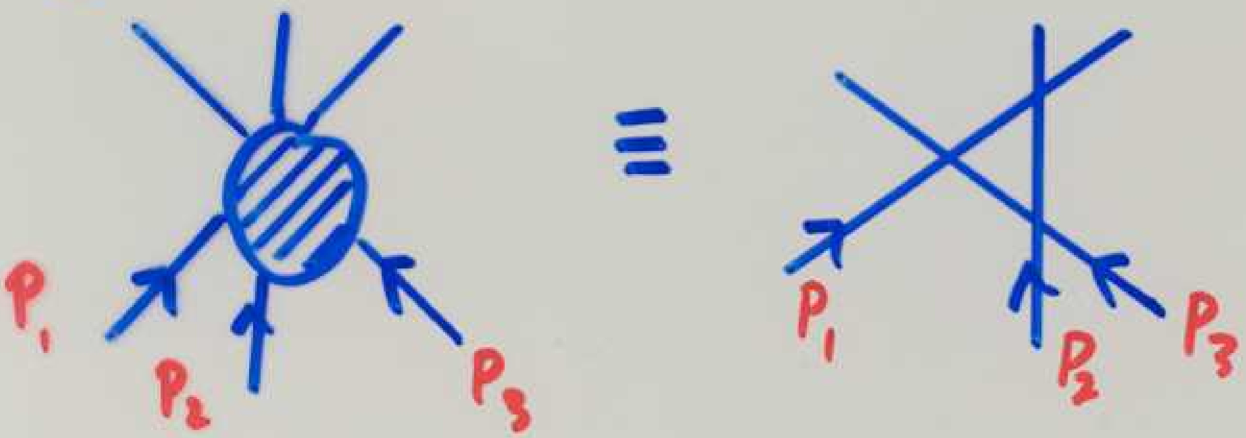
$M>2$ multi-magnon states

$$\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots\ldots$$

$$\longrightarrow p_1 \qquad \longleftarrow p_2 \qquad \longleftarrow p_3$$

integrability $\Rightarrow$ factorization of multi-particle S-matrix

eg $M=3$,



$$S(p_1,p_2,p_3) = S(p_1,p_2)\,S(p_1,p_3)\,S(p_2,p_3)$$

$\Rightarrow$ elastic scattering

conserved charges $\equiv$ individual particle momenta $\{p_1, p_2, \ldots, p_M\}$

Exact Energy Eigenstates

M-magnon scattering states $M = 1, 2, \ldots$

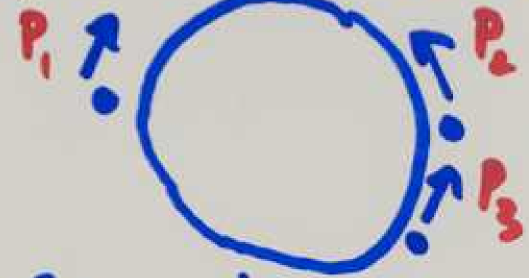
$$\psi_{p_1, \ldots, p_M}(\ell_1, \ldots, \ell_M) = \sum_{\sigma \in S_M} S_\sigma \, e^{i p_1 \ell_{\sigma(1)} + \ldots + i p_M \ell_{\sigma(M)}}$$

with energy,

$$E = \sum_{j=1}^{M} 4 \sin^2(p_j/2) \qquad - \text{①}$$

finite chain with periodic B.C.

$$\psi(\ell + L) = \psi(\ell)$$



$\Rightarrow$ Bethe Ansatz Equations

$$e^{i p_j L} = \prod_{k \neq j}^{M} S(p_j, p_k) \qquad - \text{②}$$

2 body S-matrix

Eq ① & ② determine exact spectrum

Integrability in $\mathcal{N}=4$ SUSY Yang-Mills

$G = SU(N)$

global symmetry:

$SO(4,2) \times SO(6) \subset SU(2,2|4)$

conformal R-symmetry

matter content:

A_μ gauge field

$\lambda^1_\alpha, \lambda^2_\alpha, \lambda^3_\alpha, \lambda^4_\alpha$ fermions

$\underbrace{X, Y, Z}_{\text{3 complex adjoint scalars}}$

quantum theory,

- $\beta(g^2) \equiv 0 \Rightarrow$ unbroken conformal invariance
- dual to IIB string on $AdS_5 \times S^5$ Maldacena

planar limit: 't Hooft

$N \to \infty$, $\lambda = g^2 N$ held fixed

- only planar diagrams contribute
- dual to <u>free</u> string on $AdS_5 \times S^5$

<u>observables</u> : dimension Δ of gauge-invariant operator,

$$\hat{O} \sim \mathrm{Tr}_N [X Y \mathcal{D}_\mu Z \lambda_\alpha X \dots \bar{\lambda}_{\dot{\alpha}}]$$

defined as,

$$\langle \hat{O}(x) \hat{O}(y) \rangle \sim \frac{1}{(x-y)^{2\Delta}}$$

$\Delta \equiv$ eigenvalue of dilatation generator $\mathcal{D} \in so(4,2)$

$$\Delta = \Delta_0 + \delta\Delta = \Delta_0 + \lambda \Delta_1 + \lambda^2 \Delta_2 + \dots$$

classical dimension

goal compute complete spectrum of operator dimensions $\forall \lambda$

solution Minahan + Zarembo
Beisert + Staudacher

map to spin chain, "SU(2) sector"

$$\mathrm{Tr}_N[X\,X\,Y\,X\,X\,Y\,X\,X\,Y\,X]$$
$$\uparrow\ \uparrow\ \downarrow\ \uparrow\ \uparrow\ \downarrow\ \uparrow\ \uparrow\ \downarrow\ \uparrow$$

dilatation generator $\mathcal{D}$ $\longleftrightarrow$ spin chain Hamiltonian $\hat{H}$

strong evidence for integrability of $\hat{H}$

i) Perturbation theory $\lambda \ll 1$
≤ 5 loops

ii) String theory $\lambda \gg 1$

$\Rightarrow$ Exact solution by Bethe Ansatz

• **Ground state**

chiral primary

$$\hat{O} = \mathrm{Tr}_N\,[\underset{\uparrow\;\uparrow\;\cdots\;\uparrow}{X X \ldots X}] = \mathrm{Tr}_N\,[X^J]$$

SUSY $\Rightarrow$ Exact dimension

$$\Delta = J \quad \Rightarrow \delta\Delta = 0$$

• **Magnon** Beisert

l'th site

$$\hat{O} \sim \sum_l e^{ipl} \ldots\ldots X X X \mathcal{I} X X X \ldots$$

$$\mathcal{I} \in \{Y, Z, \lambda_\alpha, D_\mu\} \leftarrow PSU(2,2|4) \text{ ``spin''}$$

exact contribution to anomalous dimension fixed by SUSY, Beisert, Dippel, Staudacher

$$\Delta - J = \sqrt{1 + \lambda/\pi^2 \sin^2(p/2)}$$

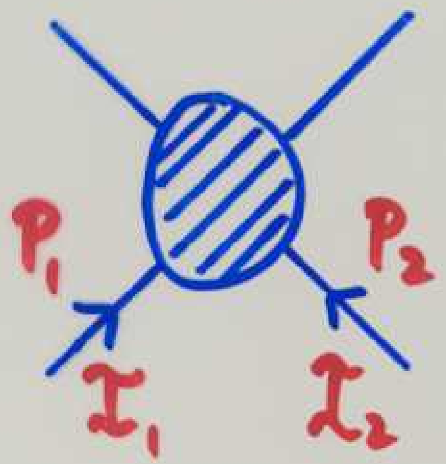
"Magnon dispersion relation"

• two-impurity operator

$\hat{O} \sim \ldots\ldots X X X I_1 X X X X X I_2 X X X X \ldots$

$\rightarrow P_1 \qquad P_2 \leftarrow$

corresponding eigenstate of $\mathcal{D}$ constructed using 2-body S-matrix



$\sim S_{I_1 I_2}(P_1, P_2; \lambda)$

Recent breakthrough Beisert
Beisert, Eden
Staudacher

$S_{I_1 I_2}$ essentially determined by SUSY, Crossing symmetry, + Lipator "transcendentality"

$\rightsquigarrow$ BES conjecture

confirmed by 4 loop calculation of Bern et. al.

- M-impurity operator

$$\hat{O} \sim \mathrm{Tr}_N [X X I_1 X X I_2 X X \ldots X X X I_M X X]$$

$$\xrightarrow{}_{p_1} \quad \xrightarrow{}_{p_2} \quad \xleftarrow{} p_M$$

integrability $\Rightarrow$ exact anomalous dimension,

$$\Delta - J = \sum_{j=1}^{M} \sqrt{1 + \lambda/\pi^2 \sin^2 \left(p_j/2 \right)} \quad - (1)$$

Bethe ansatz for SU(2) sector

$$I_1 = I_2 = \ldots = I_M = Y$$

$$e^{i p_j (J+M)} = \prod_{k \neq j}^{M} S_{YY}(p_j, p_k ; \lambda) \quad - (2)$$

known 2-body S-matrix

Eqns (1) and (2) determine exact solution for $J \gg 1$ up to corrections of $O(e^{-J})$

Conclusion + Outlook

- progress towards exact solution of $\mathcal{N}=4$ SUSY Yang-Mills in planar limit
  - Major advance in testing and understanding AdS/CFT
- outlook for QCD $N_c \to \infty$
  - integrability present in some subsectors Lipatov
  - should lead to progress in formulating weakly coupled string dual......