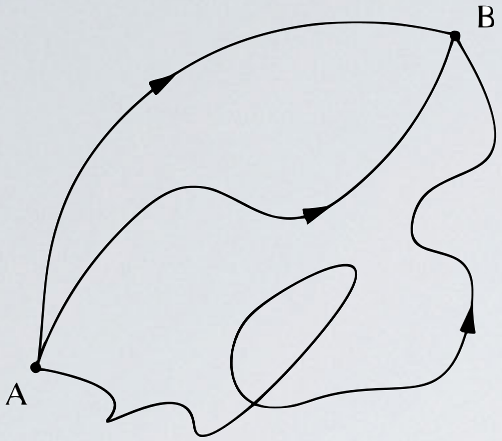

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PATH INTEGRALS, THIMBLES AND MCMC

Paul Saffin
University of Nottingham

Science & Technology Facilities Council

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

in collaboration with
Zong-Gang Mou, Anders Tranberg, Simon Woodward

1902.09147, JHEP

OVERVIEW OF TALK

- motivation

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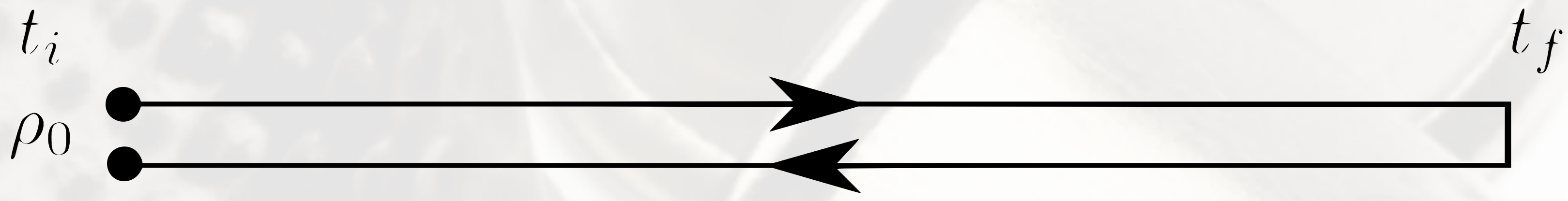
- The problem

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}\Phi \; \mathcal{O} e^{iS/\hbar}}{\int \mathcal{D}\Phi \; e^{iS/\hbar}}$$

- A solution



- what next?

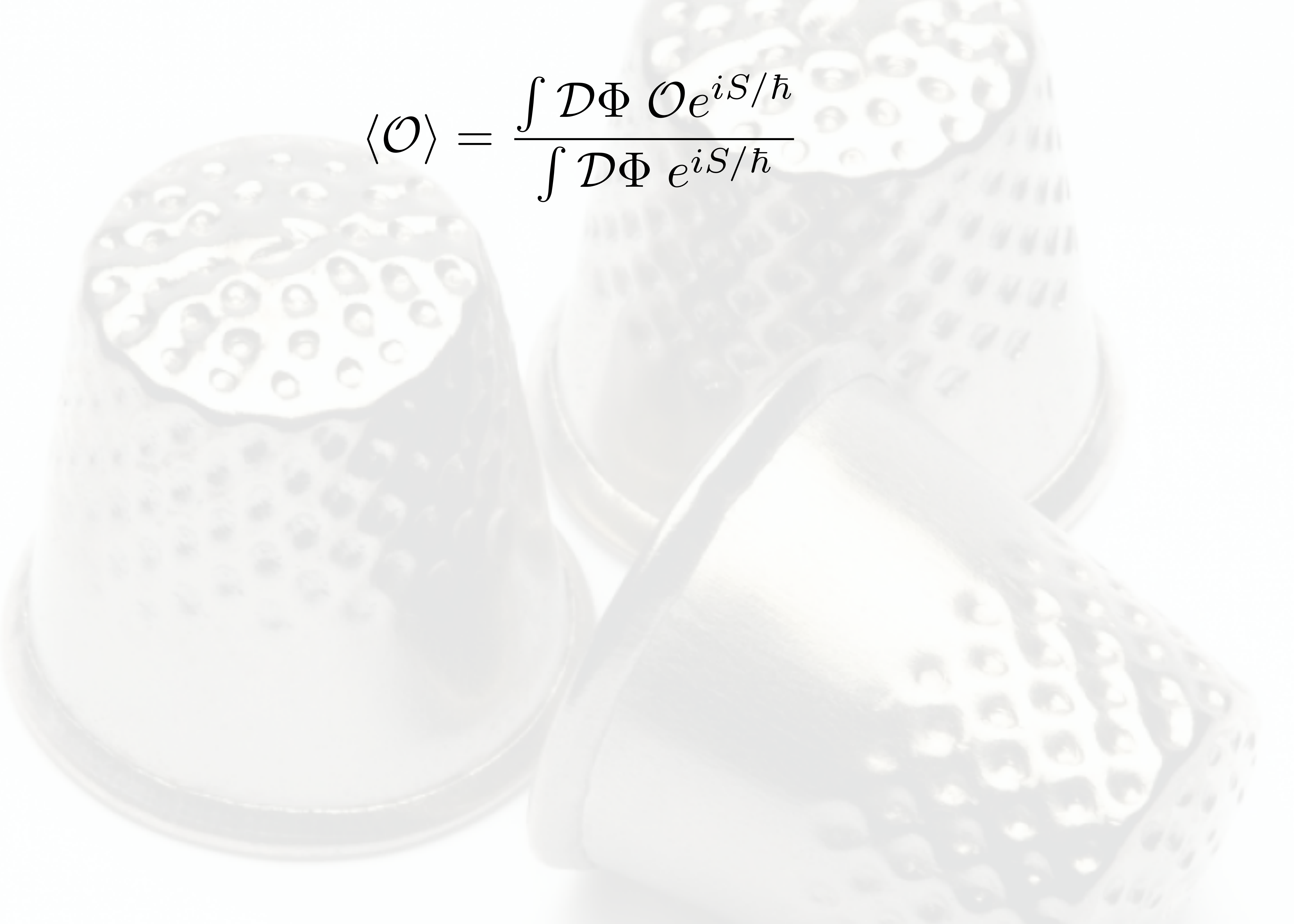
?

REAL-TIME EVOLUTION IN QUANTUM FIELD THEORY

- Preheating
  - Berges, Felder, Garcia-Bellido, Garcia-Perez,Gelfand, Gonzalez-Arroyo, Khlebnikov, Kofman, Linde, Micha, Mou, Prokopec, Pruschke, Roos, Tkachev
- Baryogenesis
  - Bodecker, Grigoriev, Hu, Kusenko, Moore, Mou, Muller, Rummukainen, PMS, Shaposhnikov,, Smit, Tranberg
- FRW evolution
  - Baacke, Covi, Heitmann, Kevlishvili, Patzold, Tranberg
- Soliton dynamics
  - Berges, Borsanyi, Hertzberg, Hindmarsh, Mou,PMS, Roth, Tognarelli Tranberg
- Thermalization
  - Aarts, Attems, Berges, Bonini, Gelfand, Kurkela, Philipsen, Pruschke, Schlicting, Sexty, Shafer, Wagenbach, Wetterich, Zafeiropoulos
- Phase transitions
  - Blaizot, Hatta, Rajantie, Sexty, Smit, Tranberg, Tsutsui
- Langevin
  - Aarts, Scherzer, Seiler, Sexty, Stamatescu

THE PATH INTEGRAL

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}\Phi \; \mathcal{O} e^{iS/\hbar}}{\int \mathcal{D}\Phi \; e^{iS/\hbar}}$$

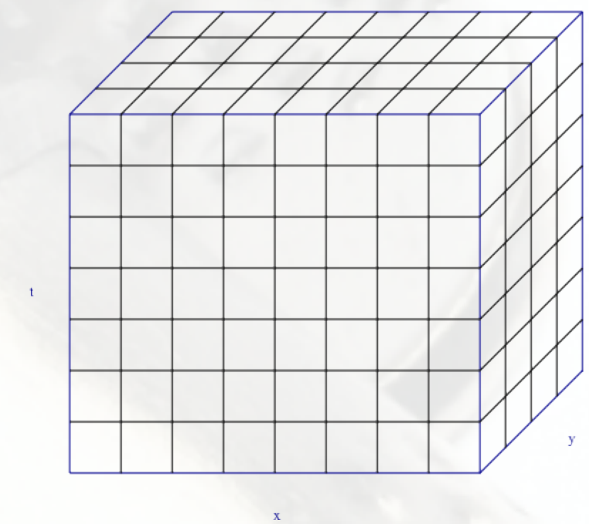


THE PATH INTEGRAL

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}\Phi \; \mathcal{O} e^{iS/\hbar}}{\int \mathcal{D}\Phi \; e^{iS/\hbar}}$$

- It's big

$$\int \mathcal{D}\Phi = \int \mathrm{d}\Phi(\mathrm{x}_1^\mu)\mathrm{d}\Phi(\mathrm{x}_2^\mu)\mathrm{d}\Phi(\mathrm{x}_3^\mu)...$$



THE PATH INTEGRAL

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}\Phi \ \mathcal{O} e^{iS/\hbar}}{\int \mathcal{D}\Phi \ e^{iS/\hbar}}$$

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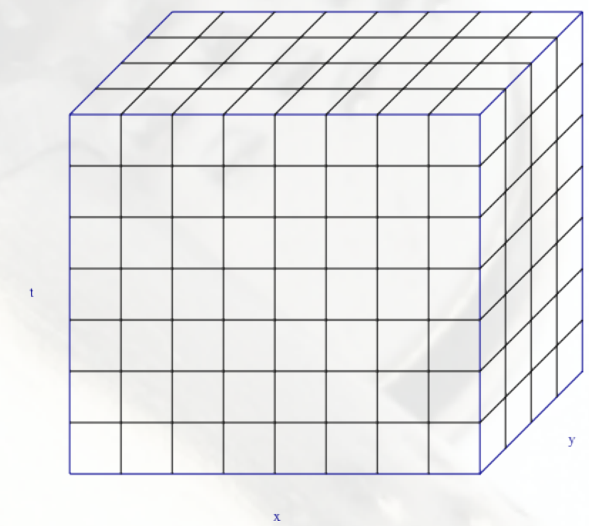
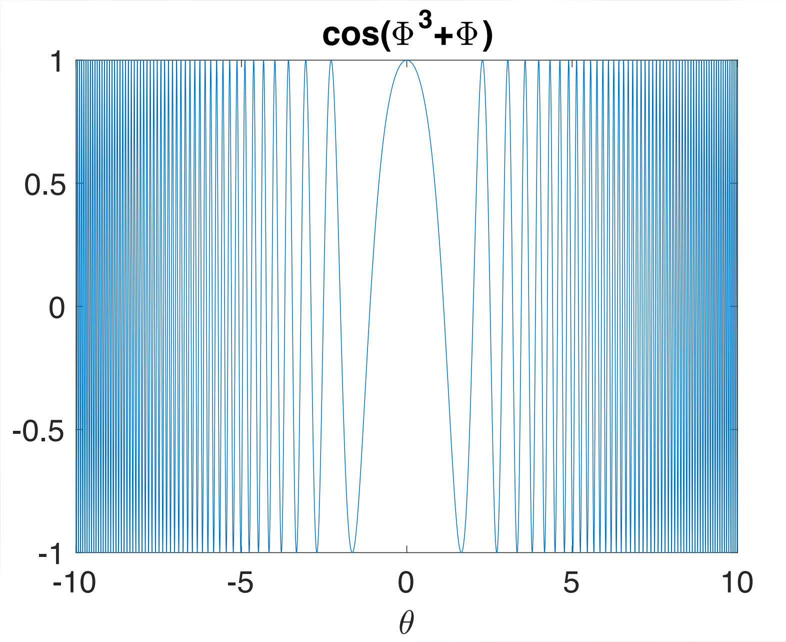


- It's a phase



THE PATH INTEGRAL

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}\Phi \ \mathcal{O} e^{iS/\hbar}}{\int \mathcal{D}\Phi \ e^{iS/\hbar}}$$

- It's big:
  - use Monte Carlo sampling to evaluate the integrals

THE PATH INTEGRAL

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}\Phi \, \mathcal{O} e^{iS/\hbar}}{\int \mathcal{D}\Phi \, e^{iS/\hbar}}$$

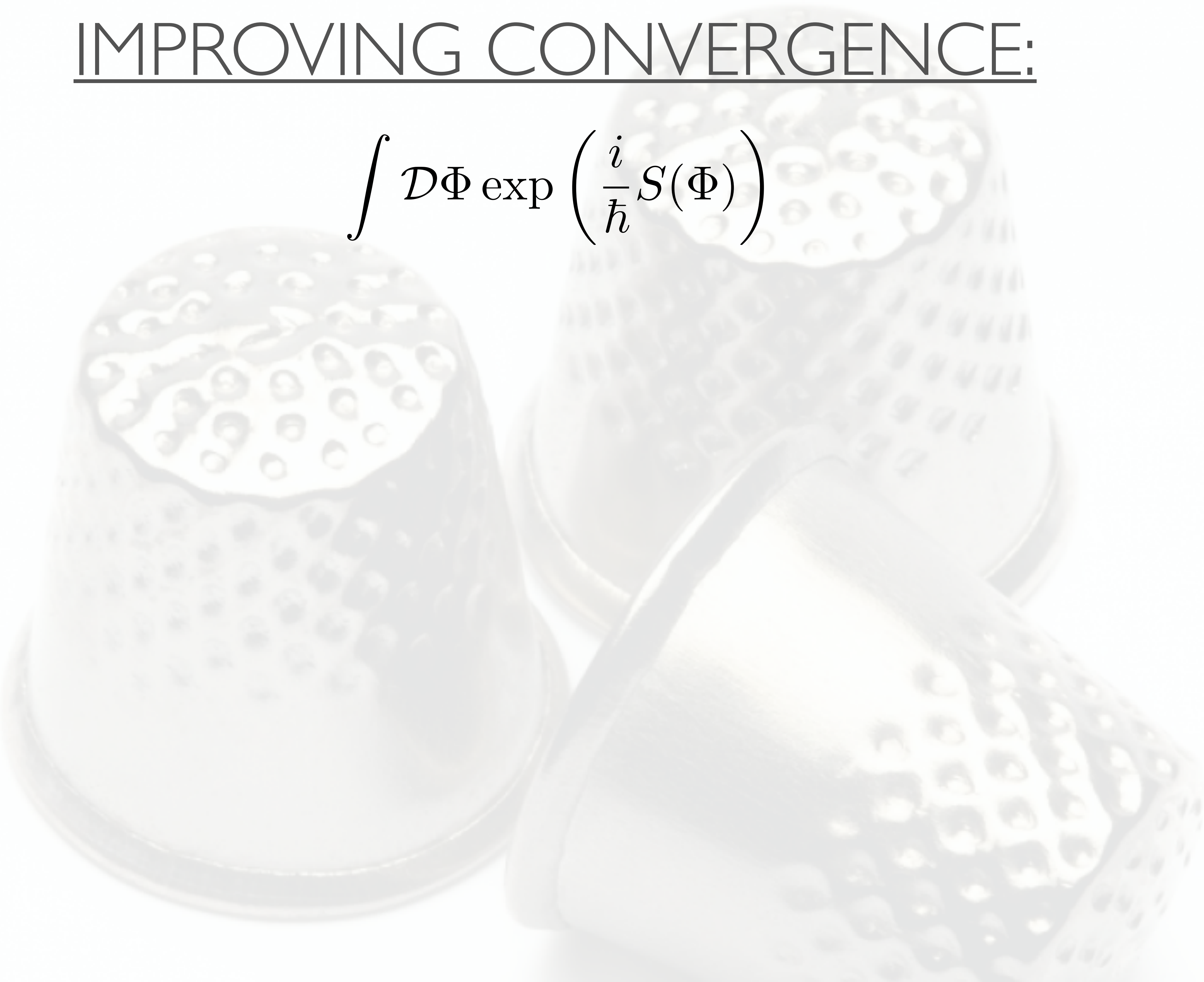
- It's big:
  - use Monte Carlo sampling to evaluate the integrals

- It's a phase:
  - use Picard-Lefschetz (Cauchy's theorem) to improve convergence

- Aarts, Alexandru, Basar, Bedaque, Christoforetti, , Feldbrugge, Fujii, Honda, Kato, Kikuwa, Komatsu, Lehners, Di Renzo, Ridgeway, Sano, Scorzato, Seiler, Sexty, Turok, Warrington, Witten,…

IMPROVING CONVERGENCE:

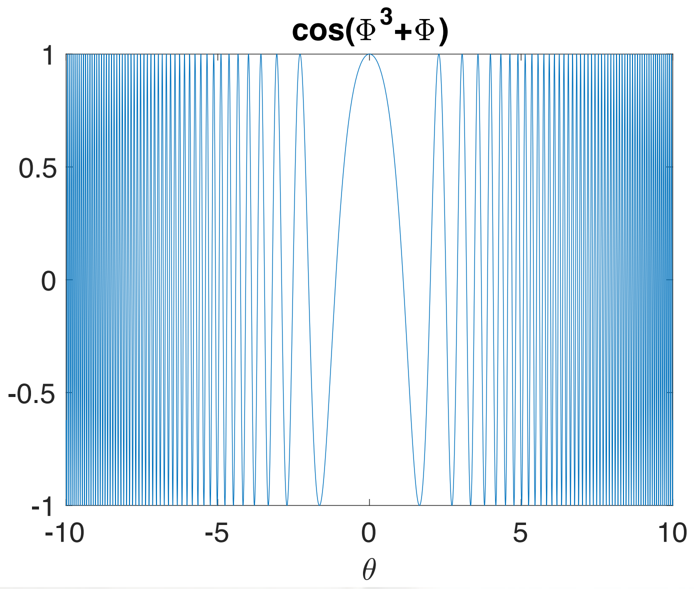
$$\int \mathcal{D}\Phi \exp\left(\frac{i}{\hbar}S(\Phi)\right)$$



IMPROVING CONVERGENCE:

$$\int \mathcal{D}\Phi \exp\left(\frac{i}{\hbar}S(\Phi)\right)$$

$$Ai(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{2\pi} \exp\left(i\left[\frac{1}{3}\phi^3 + x\phi\right]\right)$$



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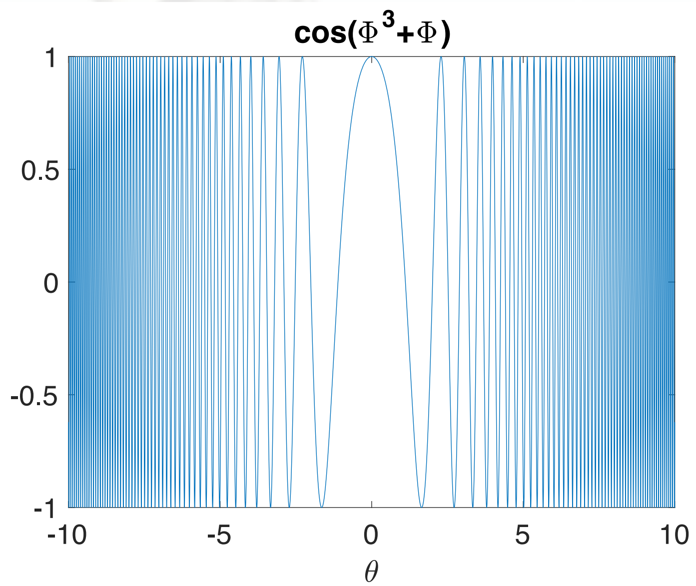
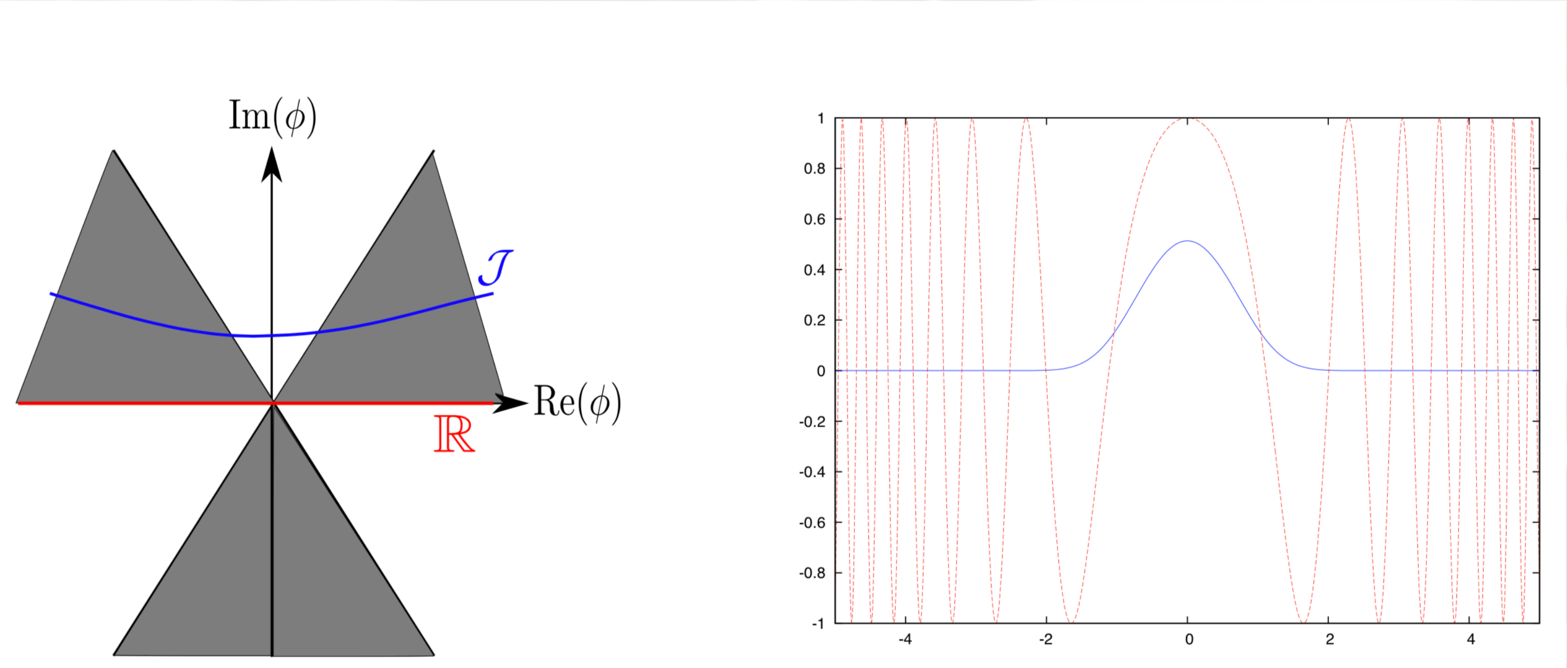


$$\phi \in \mathbb{R} \to \phi \in \mathbb{C}$$



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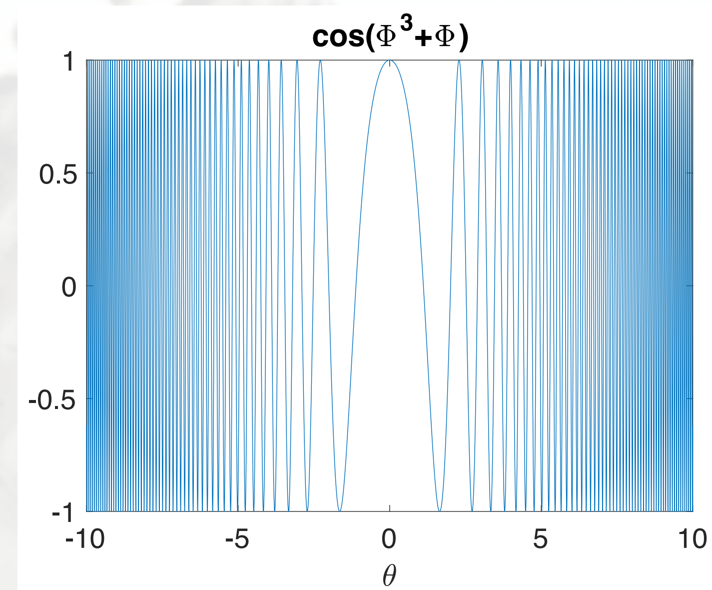
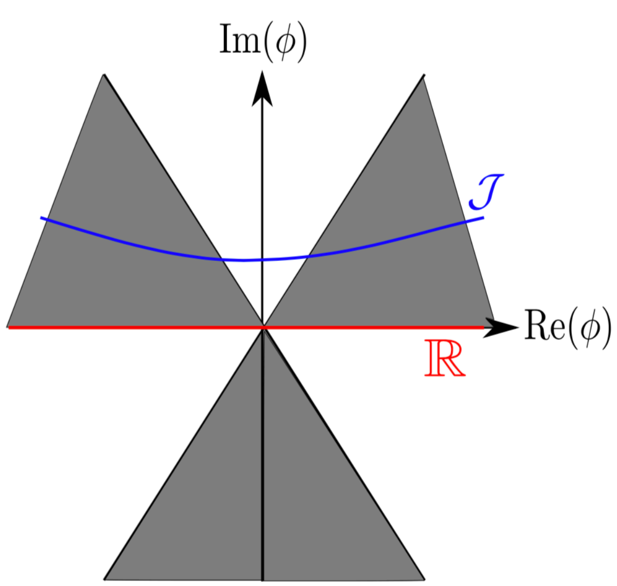
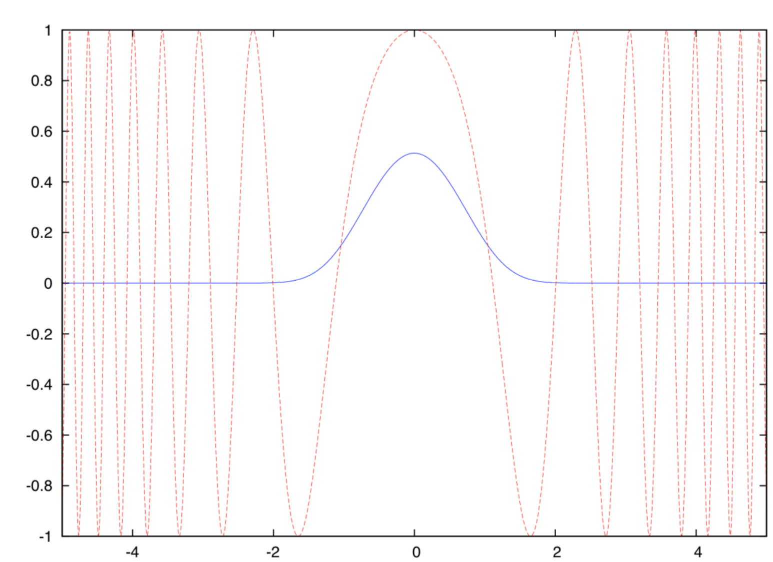


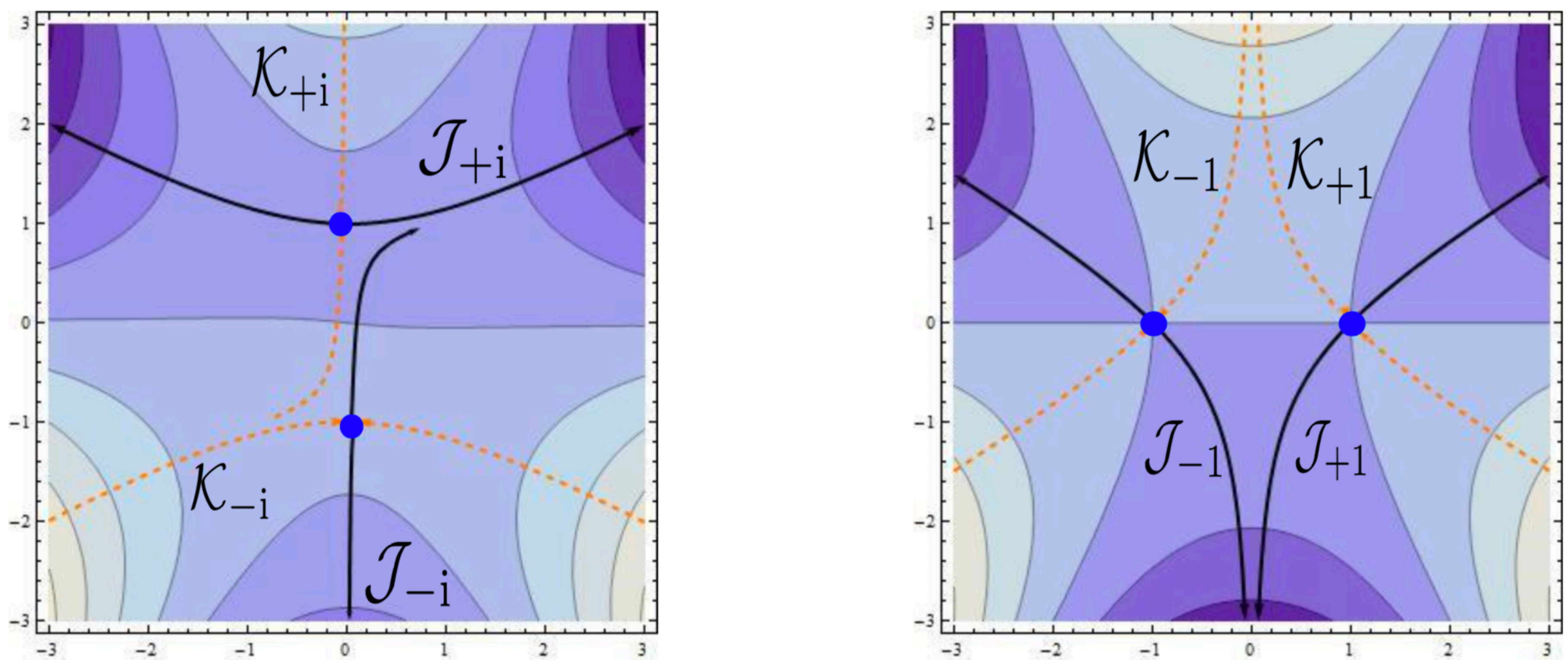
$$\phi \in \mathbb{R} \to \phi \in \mathbb{C}$$

The "best" curve is "the" thimble





FINDING "THE" THIMBLE

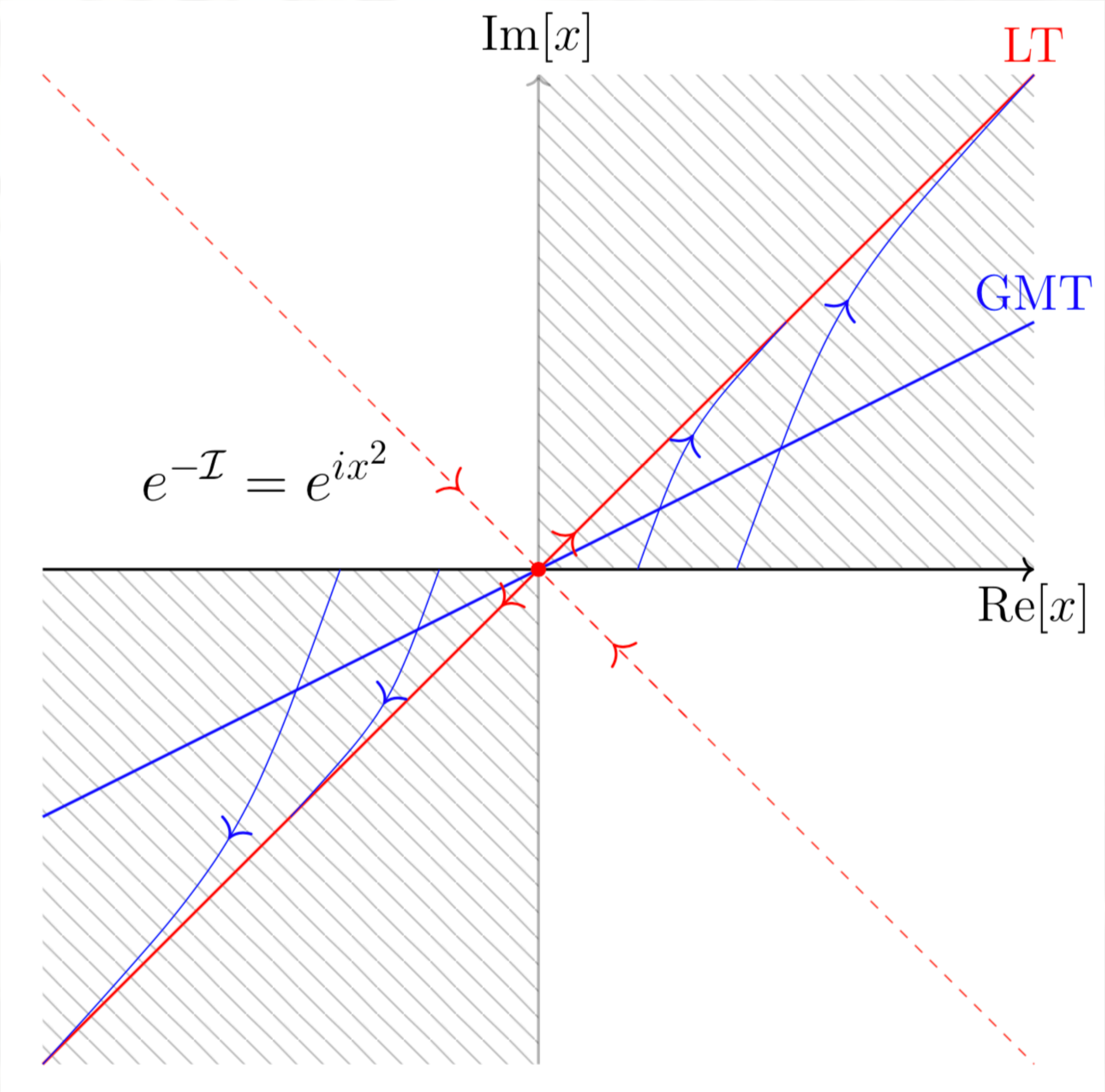


(a) $x = \mathrm{e}^{0.1\mathrm{i}}$ (b) $x = \mathrm{e}^{\pi\mathrm{i}}$

- Yuya Taniizaki

GENERALIZED THIMBLE

$$\int \mathrm{d}x\, e^{-\mathcal{I}} \quad \rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}\tau} = \overline{\left(\frac{\partial \mathcal{I}}{\partial x}\right)} \quad \rightarrow \quad \frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\tau} = \left|\frac{\partial \mathcal{I}}{\partial x}\right|^2 \geq 0$$

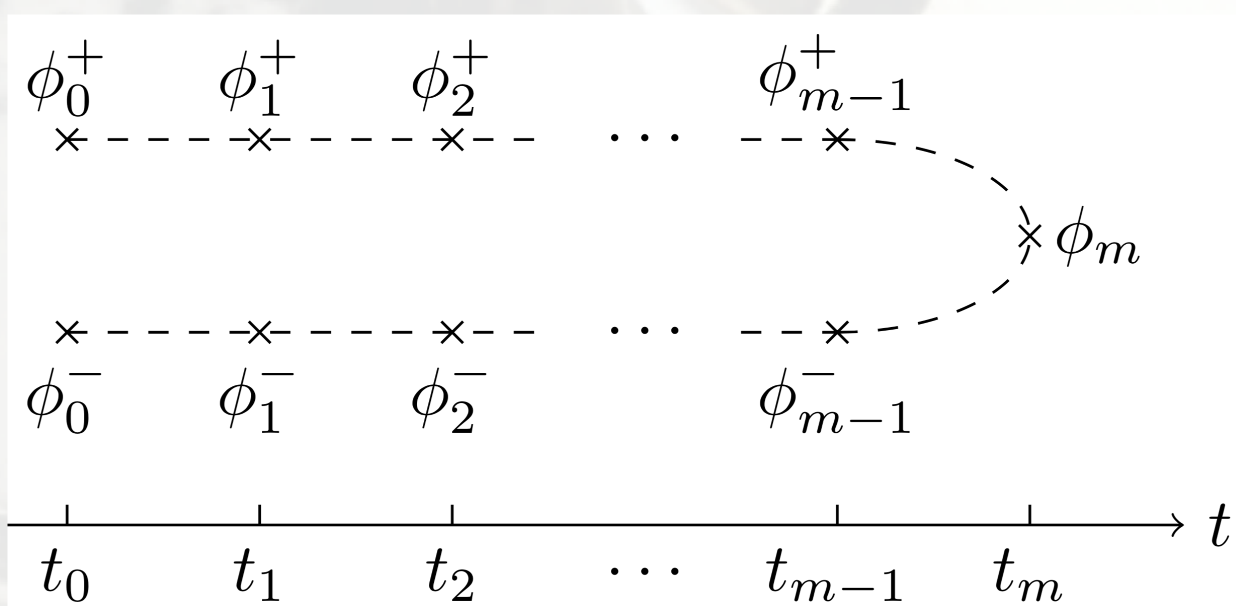


- Alexandru, Basar, Bedaque, Lamm, Lawrence.

WHAT WE WANT TO CALCULATE:

$$\langle \hat{\mathcal{O}}(t) \rangle = Tr\left(\hat{\mathcal{O}}(t)\hat{\rho}(t_0)\right) = \int \mathcal{D}\phi \langle \phi_0^-; t_0 | \hat{\mathcal{O}}(t)\hat{\rho}(t_0) | \phi_0^-; t_0 \rangle$$

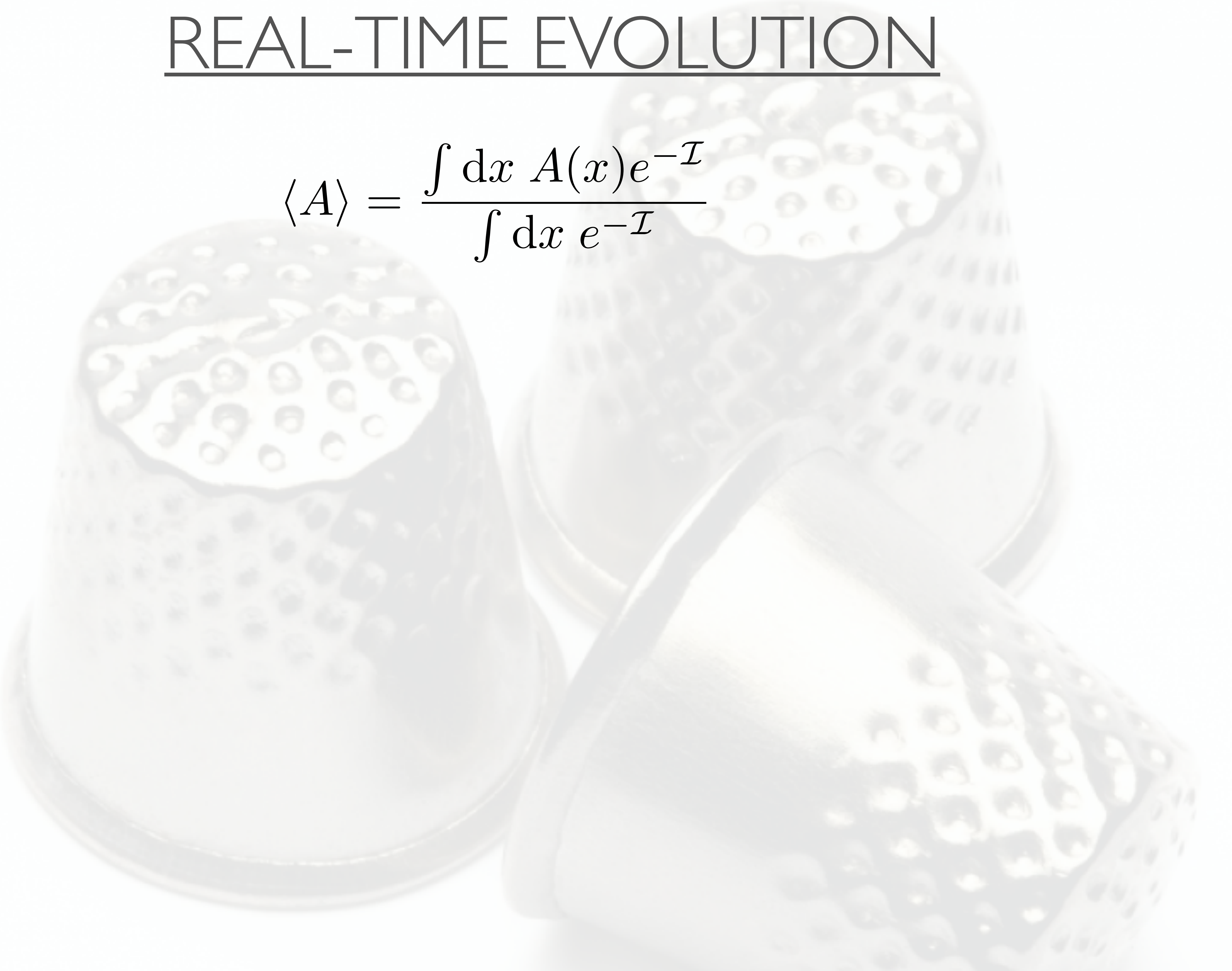
$$= \frac{\int \mathcal{D}\phi \exp\left(\frac{i}{\hbar}\int_{\mathcal{C}} \mathrm{d}t\; L\right) \mathcal{O}(t) \langle \phi_0^+; t_0 | \hat{\rho}_0 | \phi_0^-; t_0 \rangle}{\int \mathcal{D}\phi \exp\left(\frac{i}{\hbar}\int_{\mathcal{C}} \mathrm{d}t\; L\right) \langle \phi_0^+; t_0 | \hat{\rho}_0 | \phi_0^-; t_0 \rangle}$$



- Schwinger, Keldysh

REAL-TIME EVOLUTION

$$\langle A \rangle = \frac{\int \mathrm{d}x \; A(x) e^{-\mathcal{I}}}{\int \mathrm{d}x \; e^{-\mathcal{I}}}$$



REAL-TIME EVOLUTION

$$\langle A \rangle = \frac{\int \mathrm{d}x \; A(x) e^{-\mathcal{I}}}{\int \mathrm{d}x \; e^{-\mathcal{I}}}$$

Cauchy
Picard-Lefschetz

$$= \frac{\int_\Gamma \mathrm{d}z \; A(z) e^{-\mathcal{I}(z(x))}}{\int_\Gamma \mathrm{d}z \; e^{-\mathcal{I}(z(x))}}$$

REAL-TIME EVOLUTION

$$\langle A \rangle = \frac{\int \mathrm{d}x \; A(x) e^{-\mathcal{I}}}{\int \mathrm{d}x \; e^{-\mathcal{I}}}$$

cauchy

$$= \frac{\int_\Gamma \mathrm{d}z \; A(z) e^{-\mathcal{I}(z(x))}}{\int_\Gamma \mathrm{d}z \; e^{-\mathcal{I}(z(x))}}$$

co-ordinate transformation

$$= \frac{\int \mathrm{d}x \frac{\partial z}{\partial x} \; A(z(x)) e^{-\mathcal{I}(z(x))}}{\int \mathrm{d}x \frac{\partial z}{\partial x} \; e^{-\mathcal{I}(z(x))}}$$

REAL-TIME EVOLUTION

$$\langle A \rangle = \frac{\int \mathrm{d}x\ A(x) e^{-\mathcal{I}}}{\int \mathrm{d}x\ e^{-\mathcal{I}}}$$

Cauchy

$$= \frac{\int_\Gamma \mathrm{d}z\ A(z) e^{-\mathcal{I}(z(x))}}{\int_\Gamma \mathrm{d}z\ e^{-\mathcal{I}(z(x))}}$$

Co-ordinate transformation

$$= \frac{\int \mathrm{d}x \frac{\partial z}{\partial x}\ A(z(x)) e^{-\mathcal{I}(z(x))}}{\int \mathrm{d}x \frac{\partial z}{\partial x}\ e^{-\mathcal{I}(z(x))}}$$

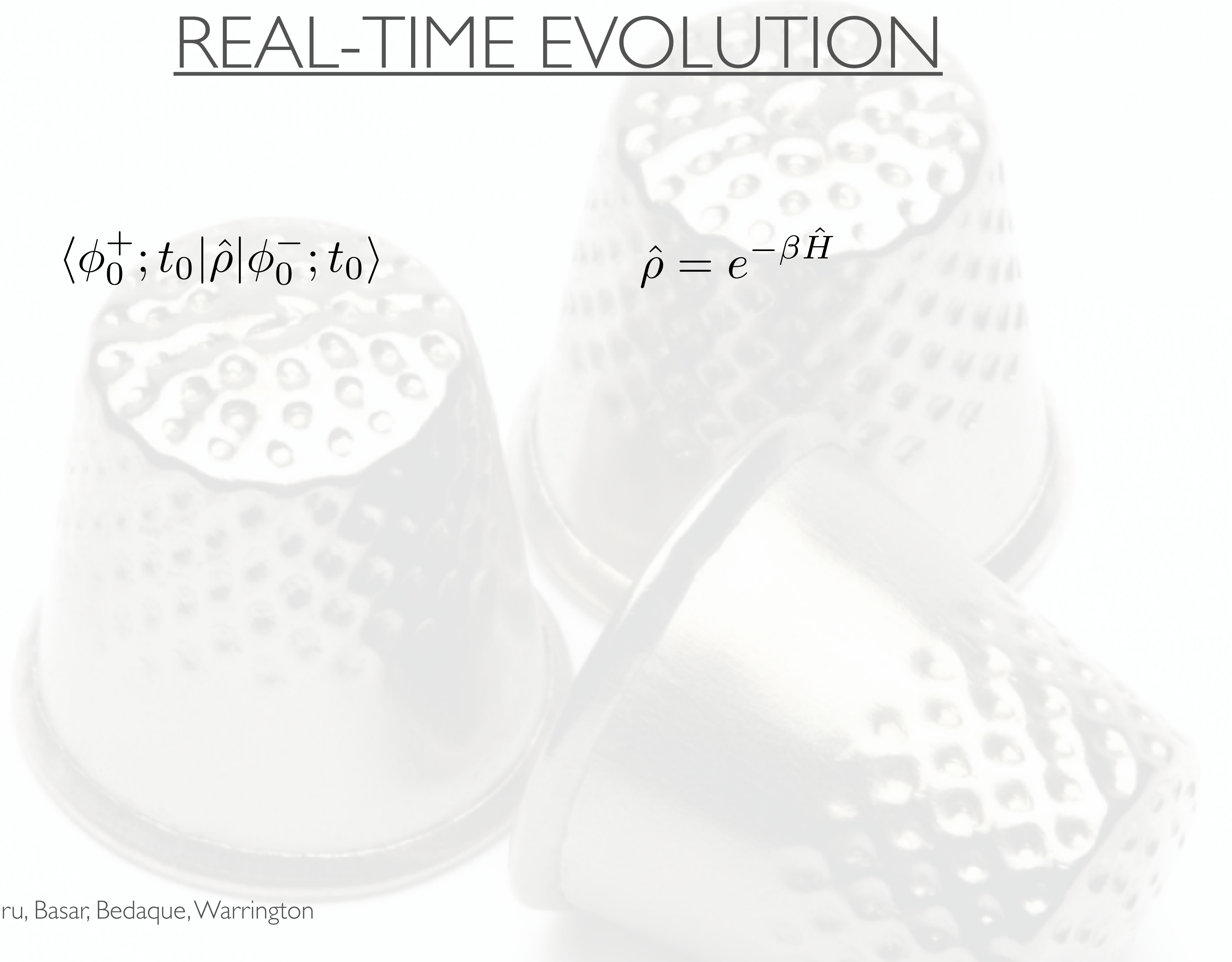
$$= \frac{\langle A\ e^{i[arg(det(J)) - I_{im}]} \rangle_P}{\langle\ e^{i[arg(det(J)) - I_{im}]} \rangle_P}$$

$$P = e^{-\mathcal{I}_{re} + \ln(|det(J)|)}$$

REAL-TIME EVOLUTION

$$\langle \phi_0^+ ; t_0 | \hat{\rho} | \phi_0^- ; t_0 \rangle$$

REAL-TIME EVOLUTION



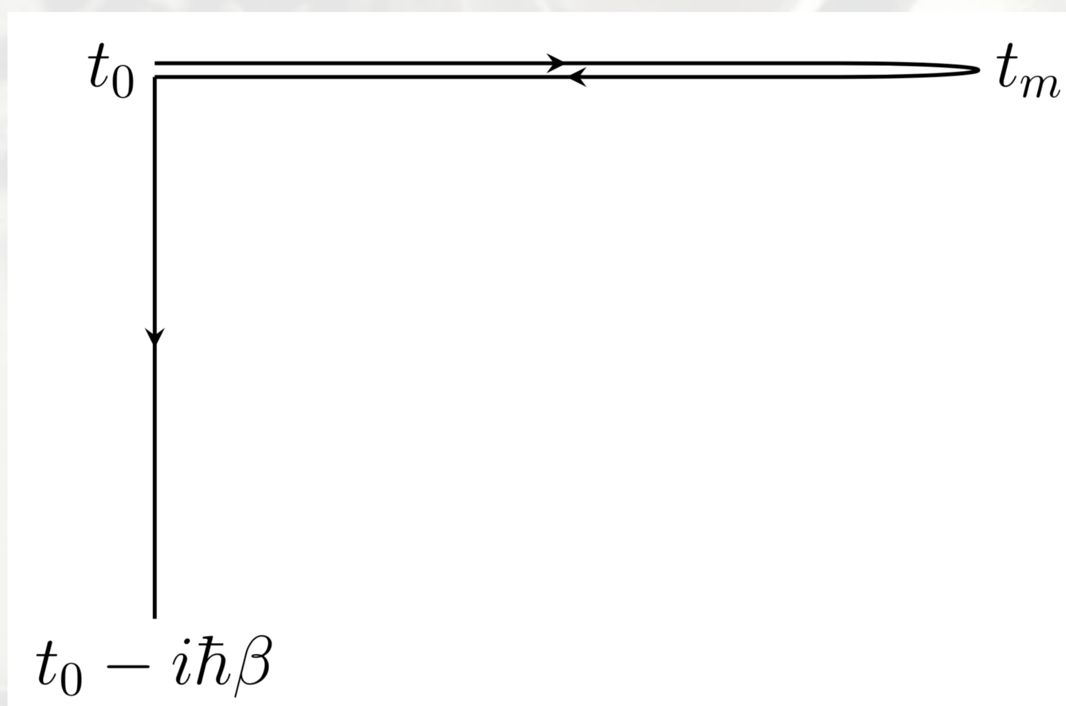
$$\langle \phi_0^+; t_0 | \hat{\rho} | \phi_0^-; t_0 \rangle \qquad \hat{\rho} = e^{-\beta \hat{H}}$$

- Alexandru, Basar, Bedaque, Warrington

REAL-TIME EVOLUTION

$$\langle \phi_0^+; t_0 | \hat{\rho} | \phi_0^-; t_0 \rangle \qquad \hat{\rho} = e^{-\beta \hat{H}}$$

$$\langle \phi_0^+; t_0 | \hat{\rho} | \phi_0^-; t_0 \rangle = \int_{\phi_0^-}^{\phi_0^+} \mathcal{D} e^{i \int_{t_0}^{t_0 - i\beta} S}$$



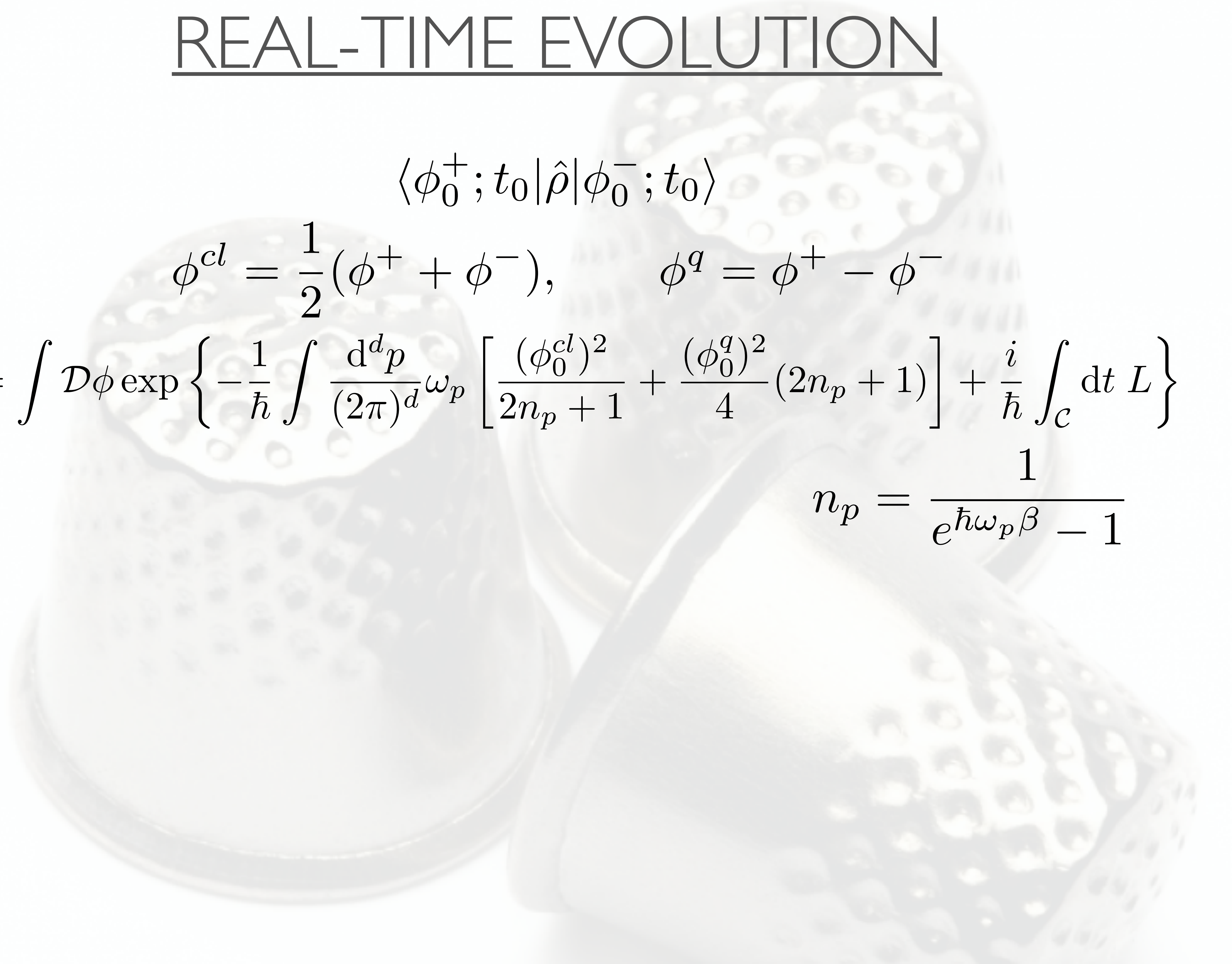
- Alexandru, Basar, Bedaque, Warrington

REAL-TIME EVOLUTION

$$\langle \phi_0^+ ; t_0 | \hat{\rho} | \phi_0^- ; t_0 \rangle$$

$$\phi^{cl} = \frac{1}{2}(\phi^+ + \phi^-), \qquad \phi^q = \phi^+ - \phi^-$$

REAL-TIME EVOLUTION



$$\langle \phi_0^+; t_0 | \hat{\rho} | \phi_0^-; t_0 \rangle$$

$$\phi^{cl} = \frac{1}{2}(\phi^+ + \phi^-), \qquad \phi^q = \phi^+ - \phi^-$$

$$Z = \int \mathcal{D}\phi \exp\left\{ -\frac{1}{\hbar} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \omega_p \left[\frac{(\phi_0^{cl})^2}{2n_p + 1} + \frac{(\phi_0^q)^2}{4}(2n_p + 1) \right] + \frac{i}{\hbar} \int_{\mathcal{C}} \mathrm{d}t \; L \right\}$$

$$n_p = \frac{1}{e^{\hbar \omega_p \beta} - 1}$$

REAL-TIME EVOLUTION

$$\langle \phi_0^+; t_0 | \hat{\rho} | \phi_0^-; t_0 \rangle$$

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$$n_p = \frac{1}{e^{\hbar \omega_p \beta} - 1}$$

$$Z = \int \mathcal{D}\phi \exp\left\{ -\frac{1}{\hbar} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \left[\omega_p \frac{(\phi_0^{cl})^2}{2n_p + 1} + \frac{(\dot{\phi}_0^{cl})^2}{\omega_p (2n_p + 1)} \right] + \frac{i}{\hbar} \int_{\mathcal{C}} \mathrm{d}t \; L \right\}$$

REAL-TIME EVOLUTION

$$\langle \phi_0^{cl}(p)\phi_0^{cl\dagger}(p')\rangle = \frac{\hbar}{\omega_p}\left(n_p + \frac{1}{2}\right)(2\pi)^d\delta(p-p')$$

$$\langle \dot{\phi}_0^{cl}(p)\dot{\phi}_0^{cl\dagger}(p')\rangle = \hbar\omega_p\left(n_p + \frac{1}{2}\right)(2\pi)^d\delta(p-p')$$

REAL-TIME EVOLUTION

$$\langle \phi_0^{cl}(p)\phi_0^{cl\dagger}(p')\rangle = \frac{\hbar}{\omega_p}\left(n_p + \tfrac{1}{2}\right)(2\pi)^d\delta(p-p')$$
$$\langle \dot{\phi}_0^{cl}(p)\dot{\phi}_0^{cl\dagger}(p')\rangle = \hbar\omega_p\left(n_p + \tfrac{1}{2}\right)(2\pi)^d\delta(p-p')$$

For a given $\left(\phi_0^{cl}, \phi_1^{cl}\right)$ there is a unique critical point $\dfrac{\delta\mathcal{I}}{\delta\phi} = 0$

$$\left(\tilde{\phi}^{cl}, \phi^q = 0\right) \qquad \left.\frac{\delta\mathcal{I}}{\delta\phi}\right|_{\tilde{\phi}^{cl}} = 0$$

One critical point

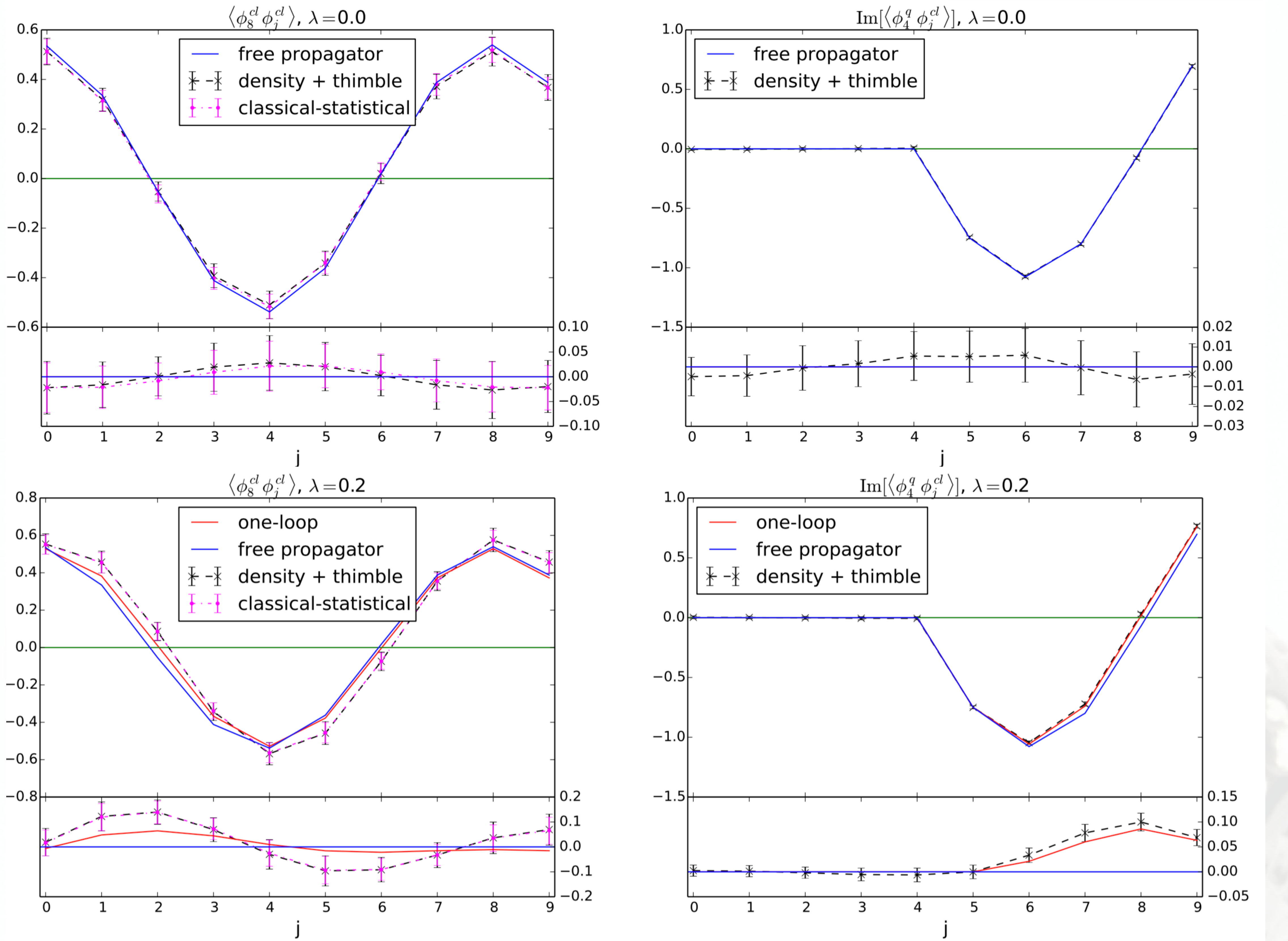
↓

One thimble

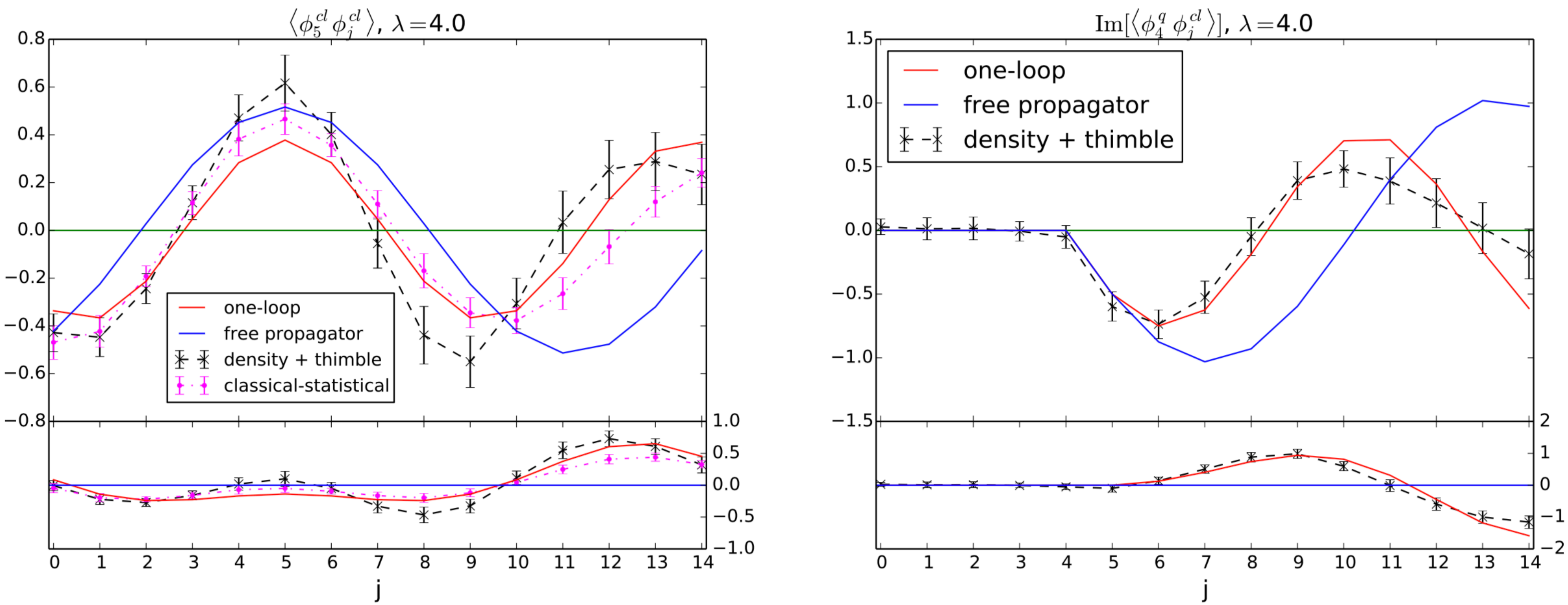
↓

Well-behaved Monte Carlo

REAL-TIME EVOLUTION



REAL-TIME EVOLUTION



CONCLUSIONS + OUTLOOK

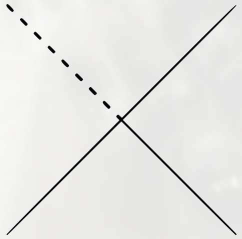
- Can solve the Lorentzian path integral using Picard-Lefschetz and Monte Carlo (scalar field)
    - Solves the sign problem
- Can we do fermions, gauge fields?
- How does this scale with dimension?
    - Is it practical for out of equilibrium studies?

CLASSICAL-STATISTICAL

$$\hbar \left(n_p + \frac{1}{2} \right) \frac{\cos(\omega_p[t_1 - t_2])}{\omega_p}$$

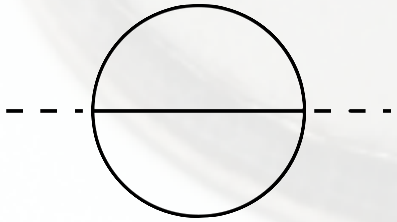
$$-i\hbar\theta(t_2 - t_1) \frac{\sin(\omega_p[t_2 - t_1])}{\omega_p}$$

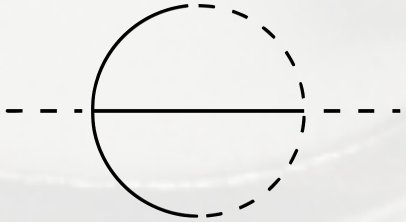
$$-i\hbar\theta(t_1 - t_2) \frac{\sin(\omega_p[t_1 - t_2])}{\omega_p}$$

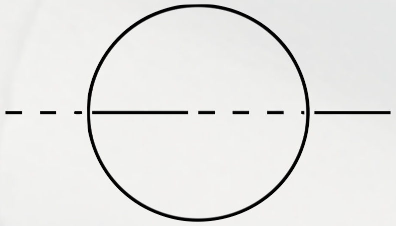


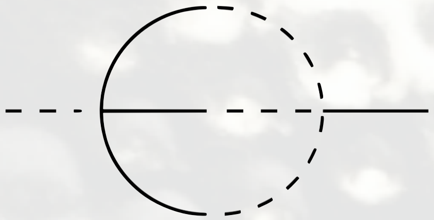
$$-\frac{i\lambda}{6\hbar}$$

$$-\frac{i\lambda}{24\hbar}$$









CRITICAL POINT

$$\hat{H} = \int \mathrm{d}^d x \left(\frac{1}{2}\pi^2 + C(\Phi) \right)$$

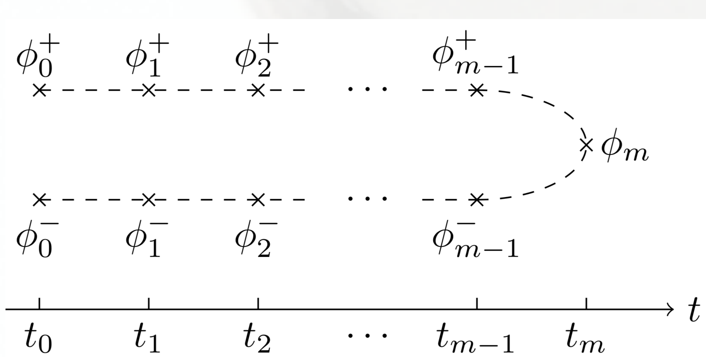
$$E_i = \int \mathrm{d}^d x \left[C\left(\phi_i^{cl}(x) + \frac{1}{2}\phi_i^q(x) \right) - C\left(\phi_i^{cl}(x) - \frac{1}{2}\phi_i^q(x) \right) \right]$$

E contains only odd powers of ϕ^q

$$\frac{\partial \mathcal{I}}{\partial \phi_i^q(x)} = -\frac{i\mathrm{d}t\mathrm{d}^d x}{\hbar} \left[\frac{2\phi_i^{cl}(x) - \phi_{i-1}^{cl}(x) - \phi_{i+1}^{cl}(x)}{\mathrm{d}t^2} - \frac{\partial E_i}{\partial \phi_i^q(x)} \right] \quad i \neq m$$

$$\frac{\partial \mathcal{I}}{\partial \phi_i^{cl}(x)} = -\frac{i\mathrm{d}t\mathrm{d}^d x}{\hbar} \left[\frac{2\phi_i^{q}(x) - \phi_{i-1}^{q}(x) - \phi_{i+1}^{q}(x)}{\mathrm{d}t^2} - \frac{\partial E_i}{\partial \phi_i^{cl}(x)} \right] \quad i \neq m$$

$$\frac{\partial \mathcal{I}}{\partial \phi_m(x)} = \frac{i\mathrm{d}^d x}{\hbar \mathrm{d}t} \phi^q_{m-1(x)}$$



REAL-TIME EVOLUTION

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