A supersymmetric model for gravity without gravitini

Pedro D. Alvarez

Manchester

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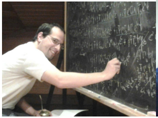
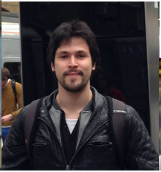
- M. Valenzuela (U. Mons, UACH) and J. Zanelli (CECs),
  d=3, $OSp(2|2)$: JHEP **1204**, 058 (2012), arXiv:1109.3944 [hep-th].
- $d = 4$, $OSp(4|2) \sim USp(2,2|1)$: P.A., Pablo Pais (U. A. Bello), J. Zanelli,
  Phys Lett B 735 (2014) 314-321, arXiv:1306.1247 [hep-th].
- d=3, $USp(2|2)$: P.A., P. Pais, E. Rodriguez (U. Concepcion),J. Zanelli,
  (in preparation).




Outline

We will present a supersymmetric model with gravity, internal gauge and matter (but without gravitini) in $d = 3$ & $4^\dagger$.

Standard supergravity multiplets $(e^a{}_\mu, \psi^\alpha_\mu, M, N, b_\mu)$:
[Stelle and West, '78,]
[Ferrara and van Nieuwenhuizen, '78].

$(e^a{}_\mu, \psi^\alpha_\mu, \cdots)$:
[Breitenlohner, '77],
[Sohnius and West, '81].

- Construction and action principle,
- Gauge invariance,
- Concluding remarks

Case $d = 3$

Gauge fields and fermionic matter in a super-connection?

The connection can be expressed more compactly as

$$\mathbb{A} = \underbrace{A\mathbb{K}}_{U(1)} + \underbrace{\overline{\mathbb{Q}}\Gamma\psi + \overline{\psi}\Gamma\mathbb{Q}}_{SUSY} + \underbrace{\omega^a\mathbb{J}_a}_{SO(2,1)} + \cancel{e^a\mathbb{P}_a}\,, \tag{1}$$

where $A = A_\mu dx^\mu$, $\omega^a = \omega^a_\mu dx^\mu = 1/2\epsilon^a{}_{bc}\omega^{bc}$ and

$$\Gamma = \gamma_\mu dx^\mu = \gamma_a e^a{}_\mu dx^\mu\,. \tag{2}$$

The nonvanishing (anti-)commutators are given by

$$[\mathbb{J}_a, \mathbb{J}_b] = \epsilon_{ab}{}^c\mathbb{J}_c\,, \tag{3}$$

$$[\mathbb{J}_a, \mathbb{Q}^\alpha] = \tfrac{1}{2}(\gamma_a)^\alpha{}_\beta\mathbb{Q}^\beta\,, \quad [\mathbb{J}_a, \overline{\mathbb{Q}}_\alpha] = -\tfrac{1}{2}\overline{\mathbb{Q}}_\beta(\gamma_a)^\beta{}_\alpha\,, \tag{4}$$

$$[\mathbb{K}, \mathbb{Q}^\alpha] = i\mathbb{Q}^\alpha\,, \qquad [\mathbb{K}, \overline{\mathbb{Q}}_\alpha] = -i\overline{\mathbb{Q}}_\alpha\,, \tag{5}$$

$$\{\mathbb{Q}^\alpha, \overline{\mathbb{Q}}_\beta\} = -(\gamma^a)^\alpha{}_\beta\mathbb{J}_a - i\tfrac{1}{2}\delta^\alpha{}_\beta\mathbb{K}\,, \tag{6}$$

where $\mathbb{J}_a = 1/4\epsilon^{ab}{}_c\mathbb{J}_{ab}$ and $\overline{\mathbb{Q}}_\alpha = (\mathbb{Q}^\alpha)^T$.

→ We do not include translations,

- Local frames $e^a{}_\mu$ connect spinors on the tangent space to the base manifold.
- The metric $g_{\mu\nu} = \eta^{ab}e^a{}_\mu e^b{}_\nu$ will be consider as dynamical (although in principle could be assumed to be fixed).

Action

In 2+1 we have the Chern-Simons action

$$S = \frac{1}{2} \int \langle \mathbb{A} d \mathbb{A} + \frac{2}{3} \mathbb{A}^3 \rangle \, . \tag{7}$$

The action is (quasi)invariant under $\mathbb{A}' = g^{-1}(\mathbb{A} + d)g$, where $g \in OSp(2|2)$.
Explicitly we have

$$S = \int AdA + \frac{1}{8}[\omega^a{}_b d\omega^b{}_a + \frac{2}{3}\omega^a{}_b \omega^b{}_c \omega^c{}_a] + \frac{1}{2}\bar{\psi}\Gamma[\overleftarrow{\nabla} - \overrightarrow{\nabla}]\Gamma\psi \, , \tag{8}$$

where $\overrightarrow{\nabla} \equiv d - iA - \frac{1}{2}\gamma_a \omega^a$, and $\overleftarrow{\nabla} \equiv \overleftarrow{d} + iA + \frac{1}{2}\gamma_a \omega^a$ are covariant derivatives for the group $U(1) \otimes SO(2,1)$ in the spin 1/2 representation.

Action

The action can be rewritten as

$$S[A,\psi,\omega,e] = \int AdA + \frac{1}{2}\left[\omega^a{}_b d\omega^b{}_a + \frac{2}{3}\omega^a{}_b\omega^b{}_c\omega^c{}_a\right] + 2\overline{\psi}\left[\overleftarrow{\partial\!\!\!/} - \overrightarrow{\partial\!\!\!/} + 2iA\!\!\!/ + \frac{1}{2}\gamma^a\omega\!\!\!/_{ab}\gamma^b\right]\psi|e|d^3x \underbrace{-2e^a T_a\overline{\psi}\psi}_{\text{mass term}}, \tag{9}$$

where $|e| = \det[e^a{}_\mu] = \sqrt{-g}$ and $T^a = de^a + \omega^a{}_b e^b$ is the torsion.

Invariance under local $U(1)$ and local $SO(2,1)$.

Extra built in symmetry: local rescaling

$$e^a(x) \to \tilde{e}^a(x) = \lambda(x)e^a(x), \qquad \psi(x) \to \tilde{\psi}(x) = \frac{1}{\lambda(x)}\psi(x). \tag{10}$$

Symmetries

An infinitesimal gauge transformation generated by

$$G = \alpha K + \frac{1}{2}\lambda^{ab} J_{ab} + \overline{Q}\epsilon - \bar{\epsilon} Q \,, \tag{11}$$

is given by

$$\delta\mathbb{A} = dG + [\mathbb{A}, G] = \delta A \mathbb{K} + \overline{\mathbb{Q}}\delta(\Gamma\psi) + \delta(\overline{\psi}\Gamma)\mathbb{Q} + \delta\omega^a \mathbb{J}_a \,, \tag{12}$$

$U(1)$:

$$\begin{aligned} \delta A &= d\alpha \\ \delta(\Gamma\psi) &= i\alpha(\Gamma\psi) \\ \delta(\overline{\psi}\Gamma) &= -i\alpha(\overline{\psi}\Gamma) \\ \delta\omega^a &= 0 \end{aligned}$$

$SO(2,1)$:

$$\begin{aligned} \delta A &= 0 \\ \delta(\Gamma\psi) &= \tfrac{1}{2}\lambda^{ab}\epsilon_{abc}\gamma^c(\Gamma\psi) \\ \delta(\overline{\psi}\Gamma) &= -\tfrac{1}{2}\lambda^{ab}\epsilon_{abc}\gamma^c(\overline{\psi}\Gamma) \\ \delta\omega^a &= d\lambda^a + \epsilon^a{}_{bc}\omega^b\lambda^c \end{aligned}$$

SUSY:

$$\begin{aligned} \delta A &= -\tfrac{i}{2}(\bar{\epsilon}\Gamma\psi + \overline{\psi}\Gamma\epsilon) \\ \delta(\Gamma\psi) &= \overrightarrow{\nabla}\epsilon \\ \delta(\overline{\psi}\Gamma) &= -\bar{\epsilon}\overleftarrow{\nabla} \\ \delta\omega^a &= -(\bar{\epsilon}\gamma^a\Gamma\psi + \overline{\psi}\Gamma\gamma^a\epsilon) \end{aligned}$$

The Lagrangian changes by a boundary term $\delta L = d\mathcal{C}_{\alpha}^{U(1)} + d\mathcal{C}_{\bar{\epsilon},\epsilon}^{susy} + d\mathcal{C}_{\lambda}^{Lor}$

$$\begin{aligned} &\mathcal{C}_{\alpha}^{U(1)} = 2\alpha dA \,, \qquad \mathcal{C}_{\bar{\epsilon}\epsilon}^{susy} = \bar{\epsilon}\overleftarrow{d}\,\Gamma\psi + \bar{\psi}\Gamma d\epsilon, \\ &\mathcal{C}_{\lambda}^{Lor} = -\tfrac{1}{2}\epsilon_{abc}\lambda^a R^{bc} + \tfrac{1}{2}(d\lambda^a + \epsilon^a{}_{bc}\omega^b\lambda^c)\omega_a \,. \end{aligned} \tag{13}$$

Field representation of the superalgebra

The variation of the composite field is $\delta(\Gamma\psi) = (\delta e^a)\gamma_a\psi + e^a\gamma_a(\delta\psi)$, where δe^a is not fixed a priori, $\mathbb{P}_a$ does not appear in the connection/algebra.

- $U(1)$ transformations, $g_\alpha = \exp[\alpha(x)\mathbb{K}]$:

$$\delta A_\mu = \partial_\mu \alpha, \quad \delta\psi = i\alpha(x)\psi, \quad \delta\overline{\psi} = -i\alpha(x)\overline{\psi}, \quad \delta\omega^a{}_\mu = 0 = \delta e^a. \tag{14}$$

- Lorentz transformations, $g_\lambda = \exp[\lambda^a(x)\mathbb{J}_a]$:

The product $\Gamma\psi = e^a\gamma_a\psi$ belongs to a reducible representation of $1 \otimes 1/2 = 1/2 \oplus 3/2$, $\delta_\lambda(\Gamma\psi) = (\delta_\lambda e^a)\gamma_a\psi + e^a\gamma_a(\delta_\lambda\psi)$, with

$$\delta_\lambda e^a = \epsilon^a{}_{bc}e^b\lambda^c, \qquad \delta_\lambda\omega^a = d\lambda^a + \epsilon^a{}_{bc}\omega^b\lambda^c \tag{15}$$

$$\delta_\lambda\psi = \frac{1}{2}\lambda^a\gamma_a\psi, \quad \delta_\lambda\overline{\psi} = -\frac{1}{2}\overline{\psi}\gamma_a\lambda^a, \quad \delta_\lambda A = 0\,. \tag{16}$$

- SUSY transformations, $g_\epsilon = \exp[\overline{\mathbb{Q}}\epsilon(x) - \overline{\epsilon}(x)\mathbb{Q}]$:

We will assume $\delta_{susy}(\gamma_\mu\psi) = \gamma_\mu\delta_{susy}\psi$. So under supersymmetry, the spin $1/2$ parts, ψ and $\overline{\psi}$, transform, while e^a remains invariant,

$$\delta A_\mu = -\tfrac{i}{2}(\overline{\psi}\gamma_\mu\epsilon + \overline{\epsilon}\gamma_\mu\psi)\,, \tag{17}$$

$$\delta\psi = \tfrac{1}{3}(\partial\!\!\!/ - iA\!\!\!/ - \tfrac{1}{2}\omega^a{}_\mu\gamma^\mu\gamma_a)\epsilon\,, \quad \delta\overline{\psi} = \overline{\delta\psi}\,, \tag{18}$$

$$\delta\omega^a{}_\mu = -(\overline{\psi}\epsilon + \overline{\epsilon}\psi)e^a_\mu - \epsilon^a{}_{bc}e^b_\mu(\overline{\psi}\gamma^c\epsilon - \overline{\epsilon}\gamma^c\psi)\,, \tag{19}$$

$$\delta e^a_\mu = 0\,. \tag{20}$$

Absence of gravitini

The invariance of the vielbein under SUSY allows to work in a linear representation,

$$\delta_\lambda(\Gamma\psi) = (\delta e^a)\gamma_a\psi + e^a\gamma_a(\delta\psi) = \nabla\epsilon\,, \tag{21}$$

$$\delta e^a{}_\mu = 0 \quad \Rightarrow \quad \delta\psi = \frac{1}{D}\not{\nabla}\epsilon\,. \tag{22}$$

But this condition also implies invariance of the metric $g_{\mu\nu} = \eta^{ab}e^a{}_\mu e^b{}_\nu$ and so the absence of gravitini.

Spin components of the Rarita-Schwinger field $(1/2 \otimes 1 = 1/2 \oplus 3/2)$:

$$\phi^\alpha_\mu = \psi^\alpha_\mu + \xi^\alpha_\mu\,, \tag{23}$$

the γ-traceless part ξ^α_μ carries the $s = 3/2$ component $(\gamma^\mu\xi^\alpha_\mu \equiv 0)$.

Projectors $P^{(1/2)} + P^{(3/2)} = 1$

$$(P^{(1/2)})_\mu{}^\nu = \frac{1}{D}\gamma_\mu\gamma^\nu\,, \quad (P^{(3/2)})_\mu{}^\nu = \delta^\nu_\mu - \frac{1}{D}\gamma_\mu\gamma^\nu\,, \tag{24}$$

$$\psi^\alpha_\mu = (P^{(1/2)})_\mu{}^\nu\phi^\alpha_\nu\,, \quad \xi^\alpha_\mu = (P^{(3/2)})_\mu{}^\nu\phi^\alpha_\nu\,, \tag{25}$$

so in our case

$$\psi^\alpha_\mu = \gamma_\mu\psi^\alpha = e^a{}_\mu\gamma_a\psi^\alpha\,, \quad \text{and} \quad \xi^\alpha_\mu \equiv 0\,, \tag{26}$$

Absence of gravitini

We act with the projectors on the eq.

$$\delta_\lambda(\Gamma\psi) = (\delta e^a)\gamma_a\psi + e^a\gamma_a(\delta\psi) = \nabla\epsilon\,, \qquad (27)$$

that tell us $\psi = \frac{1}{D}\not\nabla\epsilon$ and force us to impose the condition

$$P_\nu^{(3/2)\,\mu}\nabla_\mu\epsilon = 0\,, \qquad (28)$$

Thus

$$\nabla_\mu\epsilon = \gamma_\mu\chi(x) \quad \Rightarrow \quad \delta\psi = \chi(x)\,, \qquad (29)$$

Integrability conditions $[\nabla_\mu, \nabla_\nu]\epsilon \Rightarrow$

Flat space: $\epsilon = \epsilon^{(0)} + x^\mu\gamma_\mu\epsilon^{(1)}$ where $\epsilon^{(0)}, \epsilon^{(1)} = \mathit{const}$,
Flat space & $A_\mu = \partial_\mu\alpha(x)$: $\epsilon = e^{i\alpha}(\epsilon^{(0)} + x^\mu\gamma_\mu\epsilon^{(1)})$.

Here we will comment on the existence of nontrivial classical solutions. For the present picture they are relevant as a dynamical symmetry breaking mechanism.

The field equations are

$$\delta A \quad \rightarrow \quad F_{\mu\nu} = \epsilon_{\mu\nu\lambda} j^{\lambda} \,, \quad j^{\mu} = -i|e|\overline{\psi}\gamma^{\mu}\psi \,, \tag{30}$$

$$\delta\omega \quad \rightarrow \quad R^{ab} = 2e^{a}e^{b}\overline{\psi}\psi \,, \quad \Rightarrow \quad R^{a}{}_{b}e^{b} = 0 = DDe^{a} = DT^{a} \,, \tag{31}$$

$$\delta\overline{\psi} \quad \rightarrow \quad \left[\partial\!\!\!/ - i A\!\!\!/ - \frac{1}{4}\gamma^{a}\omega\!\!\!/_{ab}\gamma^{b} + \frac{\kappa}{2} + \frac{1}{2|e|}\partial_{\mu}(|e|E_{a}{}^{\mu})\gamma^{a} \right] \psi = 0 \tag{32}$$

$$\delta e \quad \rightarrow \quad \overline{\psi}\left[\gamma^{b}\Delta^{\mu\lambda}_{ab}\overrightarrow{\partial\!\!\!/}_{\lambda} - \overleftarrow{\partial\!\!\!/}_{\lambda}\gamma^{b}\Delta^{\mu\lambda}_{ab} - 2i\gamma^{b}\Delta^{\mu\lambda}_{ab}A_{\lambda} + \epsilon^{\mu\nu\lambda}T_{a\nu\lambda} \right]\psi = 0 \,, \tag{33}$$

where

$$|e|\kappa d^{3}x \equiv e^{a}T_{a} \,, \quad \text{and} \quad \Delta^{\mu\nu}_{ab} = |e|(E_{a}{}^{\mu}E_{b}{}^{\nu} - E_{b}{}^{\mu}E_{a}{}^{\nu}) \,. \tag{34}$$

Classical Solutions

Let us consider infinitesimal fermionic exitatons $\psi \sim \varepsilon$

$$F_{\mu\nu} = 0\,, \tag{35}$$
$$R^{ab} = 0 = d\omega^{ab} + \omega^{a}{}_{c}\omega^{cb}\,, \tag{36}$$

By counting free components we can suggest the following ansatz

$$T^a = \tau\epsilon^{abc}e_b e_c + \beta e^a\,, \qquad \overset{DT^a=0}{\Longrightarrow} \qquad d\tau + \tau\beta = 0\,, \quad d\beta = 0\,, \tag{37}$$

we have either (I) $\tau = 0$ and β-closed or (II) $\tau \neq 0$ and $\beta = -d\log\tau$, but (II) contains (I), so we can chose

$$T^a = \tau\epsilon^{abc}e_b e_c - \frac{d\tau}{\tau}e^a\,, \tag{38}$$

and using the Weyl invariance we can finally write

$$T^a = -\frac{m}{3}\epsilon^{abc}e_a e_b\,. \tag{39}$$

The integration constant m can be identified as the mass of the fermionic excitation.

Constant Curvature Solutions

We separate the metric contribution to the torsion

$$\omega^{ab} = \overline{\omega}^{ab} + \kappa^{ab}\,, \quad de^a + \overline{\omega}^a{}_b e^b = 0\,, \quad T^a = \kappa^a{}_b e^b\,, \tag{40}$$

where $\kappa^{ab} = -\kappa^{ba}$ is the contorsion.
From the solution of the torsion we read the contorsion

$$T^a = -\frac{m}{3}\epsilon^{abc} e_a e_b = \kappa^a{}_b e^b \quad \Rightarrow \quad \kappa_{ab} = \frac{m}{3}\epsilon_{abc} e^c\,, \tag{41}$$

using this we obtain an expresion for the Riemann tensor

$$R^{ab} = \overline{R}^{ab} + \underbrace{\overline{D}\kappa^{ab}}_{\stackrel{\shortparallel}{0}} + \kappa^a{}_c \kappa^{cb} = \overline{R}^{ab} + \frac{2}{9} m^2 e^a e^b = 0\,, \tag{42}$$

where we recognize the cosmological constant as $\lambda = -2m^2/9$.

The values of the mass and the cosmological constant are linked.

Constant Curvature Solutions

Solutions of constant curvature are well known [Brown and Henneaux, 1986], [Banados, Teitelboim and Zanelli, 1992].

Under circular symmetry we have

$$ds^2 = -f^2 dt^2 + f^{-2} dr^2 + (r d\phi - N dt)^2 , \tag{43}$$

$$f^2 = (r/\ell)^2 - M + (J/2r)^2 , \quad N = -J/2r^2 , \quad \lambda = -2/\ell \tag{44}$$

BTZ	$M\ell > \vert J \vert$
extremal BTZ	$M\ell = \vert J \vert$
AdS	$J = 0$ and $M = -1$
naked conical singularity	$-\vert J \vert < M\ell < 0$

Killing spinor solutions exist for AdS, massless BTZ and the extremal BTZ case preserving all, half and 1/4 of the supersymmetries respectively [Coussaert and Henneaux, 1994].

The existence of killing spinors implies that the bosonic BPS vacua is stable in SG.

Case $d = 4$

Connection for $OSp(2|4)$

In $d = 4$ we use $USp(2, 2|1)$. Translations must be included. in the connection:

$$\mathbb{A} = A\mathbb{K} + \overline{\mathbb{Q}}_{\alpha}\Gamma\psi^{\alpha} + \overline{\psi}_{\alpha}\Gamma Q^{\alpha} + f^{a}\mathbb{J}_{a} + \frac{1}{2}\omega^{ab}\mathbb{J}_{ab}\,, \tag{45}$$

where $a = 0, \cdots, 3$, $\alpha = 1, \cdots, 4$. The curvature is given by

$$\mathbb{F} \equiv d\mathbb{A} + \mathbb{A}\wedge\mathbb{A} = \mathcal{F}\mathbb{K} + \overline{\mathbb{Q}}_{\alpha}\mathcal{F}^{\alpha} + \overline{\mathcal{F}}_{\alpha}Q^{\alpha} + \mathcal{F}^{a}\mathbb{J}_{a} + \frac{1}{2}\mathcal{F}^{ab}\mathbb{J}_{ab}\,, \tag{46}$$

What invariants we can use as an action principle?

so

$$\mathbb{F} \sim F\mathbb{K} + \overline{\mathbb{Q}}^{i}_{\alpha}\mathcal{F}^{\alpha}_{i} + Df^{a}\mathbb{J}_{a} + \frac{1}{2}R^{ab}\mathbb{J}_{ab}\,, \tag{47}$$

The only invariant is

$$P_1 = \langle\mathbb{F}\mathbb{F}\rangle \tag{48}$$

The invariant P_1 is a closed form –the Chern class–, whose integral over a compact manifold is a topological invariant (Chern-Weil theorem).

The Lagrangian must be an invariant of smaller group.

Sensible invariants engineering

MacDowell, Mansouri, Phys Rev Lett 38 (1977) 739-742,
Chamseddine, West, Nucl Phys B 129 (1977) 39.

For the gravity part we need a symmetry breaking operator. Using $S^A{}_B = (\Gamma_5)^A{}_B$ we define

$$\tilde{\mathbb{F}} = *\mathcal{F}\mathbb{K} + \overline{\mathbb{Q}}_\alpha \mathcal{F}^\alpha + \overline{\mathcal{F}}_\alpha Q^\alpha + \mathcal{F}^a \mathbb{J}_a + \frac{1}{2}\mathcal{F}^{ab}\mathbb{J}_{ab} \,, \tag{49}$$

possible invariants are

$$P_1 = \langle \mathbb{F} \wedge \mathbb{F} \rangle \,, \quad P_2 = \langle \mathbb{F} \wedge *\mathbb{F} \rangle \,, \quad P_3 = \langle \mathbb{F} \wedge \tilde{\mathbb{F}} \rangle \,, \tag{50}$$

$$P_4 = \langle S.\mathbb{F} \wedge \mathbb{F} \rangle \,, \quad P_5 = \langle S.\mathbb{F} \wedge *\mathbb{F} \rangle \,, \quad P_6 = \langle S.\mathbb{F} \wedge \tilde{\mathbb{F}} \rangle \,. \tag{51}$$

- P_1 is a topological invariant.
- P_4 does not yield a Lagrangian for the $U(1)$ field.
- P_3 and P_5 give gravitational Pontryagin forms.
- P_2 have second order derivatives for the fermion.
- P_6 give better results.

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The Lagrangian will be $L = \langle F \circledast F \rangle$, where $\circledast = (*, S)$ $(\Rightarrow \circledast^2 = -1)$

Action for $d = 4$

Gauge and gravity kinetic terms:

$$L \supset 2F * F = |e| F_{\mu\nu} F^{\mu\nu} d^4 x \,, \tag{52}$$

$$L \supset \frac{1}{4}\epsilon_{abcd}(R^{ab} + f^a f^b)(R^{cd} + f^c f^d)\,. \tag{53}$$

$$L \supset \overline{\psi}\not{e}\gamma_5 \not{f}\hat{\nabla}(\not{e}\psi) - (\overline{\psi}\not{e})\overleftarrow{\hat{\nabla}}\not{f}\gamma_5\not{e}\psi \,, \tag{54}$$

'Townsend' identification $f^a = \mu e^a$ Phys Rev D 15 (1977) 2795

Nambu-Jona-Lasinio term for dynamical symmetry breaking
Phys. Rev. 122 (1961) 345; Phys. Rev. 124 (1961)

$$L \supset g[(\bar{\psi}\psi)^2 - (\bar{\psi}\Gamma_5\psi)^2]\,. \tag{55}$$

Action for $d = 4$

Scales come with the identification $f^a = \mu e^a$ and $\psi_{\text{physical}} \sim \sqrt{\nu}\psi$.
Fermion cuadratic mass term: $m \sim \mu^2/\nu$ and NJL coupling constant: $g = (3\nu)^{-2}$.
Newton's constant $G = -s^2(4\pi\mu^2)^{-1}$ and cosmological constant $\Lambda = -s^2\mu^2$.
NJL mass for a cut-off $\mathcal{M}$,

$$\frac{m_{\text{NJL}}^2}{\mathcal{M}^2} \log\left[1 + \frac{\mathcal{M}^2}{m_{\text{NJL}}^2}\right] = 1 - \frac{2\pi^2}{g\mathcal{M}^2}. \tag{56}$$

Contributions to the cosmological constant,

$$\Lambda_{\text{eff}} = \Lambda + \frac{2}{\nu}\langle\overline{\psi}\psi\rangle - \frac{3m_{\text{NJL}}}{2\mu^2}\langle\overline{\psi}\psi\rangle\,, \tag{57}$$

Is it possible to avoid fine tunning?.

Summary

- Local $U(1)$ and $SO(2,1)$ and SUSY if the background allow it.
- The metric is required by matter ($s = 1/2$).
- Mass splitting without or with partial susy breaking.
- Weyl invariance $e \to \lambda e$. Mass term without breaking conformal symmetry.
- Existence of classical solutions.

What follows

- Level of fine tunning.
- Cosmological applications.
- Chiral matter.
- Higher dimensions.

Thank you for your attention!

Backup slides

SUSY hallmarks

- Only non-trivial unification of Poincaré and internal symmetries.
- Fewer free parameters / hierarchy problem.
- Positivity of energy, stable groundstates (BPS).
- Improved U.V. behaviour $\infty_B + \infty_F = 0$.
- Unification betwen B-F.

$$\begin{bmatrix} B \\ F \end{bmatrix}' = Q \begin{bmatrix} B \\ F \end{bmatrix} \tag{58}$$

We need SUSY-Breaking!.

Bosons and Fermions in standard model

Bosons	Fermions
Carriers of interactions	Building blocks of matter
Interaction potentials	Sources
(not conserved)	(conserved currents)
Spin 1 fields (poss. ex. Higgs)	Spin 1/2
1-forms $A_\mu dx^\mu$	zero-forms ψ
Connections (adj. rep.)	sections (vector reps.)
2nd order field eqns.	1st order field eqns.

Supersymmetry trick

For each field include another of the opposite statistics

photon	$\rightarrow$	photino	electron	$\rightarrow$	selectron
gluon	$\rightarrow$	gluino	quark	$\rightarrow$	squark
graviton	$\rightarrow$	gravitino	neutrino	$\rightarrow$	sneutrino
boson	$\rightarrow$	bosino	fermion	$\rightarrow$	sfermion

Bosons and Fermions in a connection

A good suggestion come from the similarity of kinetic terms of a Chern-Simons theory and a Dirac spinor in 3-dimensions:

$$AdA\,, \qquad \overline{\psi}\partial\!\!\!/\psi\,, \tag{59}$$

In fact, by defining:

$$\mathbb{A} = \begin{bmatrix} A\!\!\!/ & \psi \\ \overline{\psi} & 0 \end{bmatrix} = \begin{bmatrix} A\!\!\!/^{\alpha}{}_{\beta} & \psi^{\alpha} \\ \overline{\psi}_{\beta} & 0 \end{bmatrix}_{3\times 3}, \tag{60}$$

we get the correct transformation laws,

$$g = \begin{bmatrix} e^{i\alpha(x)} & 0 & 0 \\ 0 & e^{i\alpha(x)} & 0 \\ 0 & 0 & e^{2i\alpha(x)} \end{bmatrix} = \exp[\alpha(x)\mathbb{K}]\,, \tag{61}$$

$$\mathbb{A} \to \mathbb{A}' = g^{-1}\mathbb{A}\ g + g^{-1}d\!\!\!/ g \quad \Rightarrow \quad \begin{cases} A' = g^{-1}A\ g + g^{-1}dg \\ \psi' = g^{-1}\psi \\ \overline{\psi}' = \overline{\psi}\ g \end{cases}, \tag{62}$$

where $\mathbb{K} = i\ \mathrm{diag}(1,1,2)$ and $d\!\!\!/ = \gamma^{\mu}\partial_{\mu}$.

We could consider now $g \in U(1) \subset G \leftarrow$ supergroup.

Antisymmetric γ-product

Let us consider the a set of γ matrices $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, we can define a 1-form in the exterior algebra defined by antisimetrized product of γ matrices

$$A = A_\mu dx^\mu \qquad \longleftrightarrow \qquad \not{A} = A_\mu \gamma^\mu \tag{63}$$

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu \qquad \longleftrightarrow \qquad \gamma^\mu \tilde{\wedge} \gamma^\nu \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] = -\gamma^\nu \tilde{\wedge} \gamma^\mu \tag{64}$$

$$d^2 = 0 \qquad \longleftrightarrow \qquad \not{d}^2 = 0 \tag{65}$$

The γ^μ matrices span a basis for an exterior algebra defined by the anti-symmetrized product $\tilde{\wedge}$.

A more standard expression is obtained by writing $\mathbb{A} = \mathbb{A}^a_\mu \mathbb{T}_a dx^\mu$, $\mathbb{T}_a \in osp(2|2)$

$$\mathbb{A}_\mu = A_\mu \mathbb{K} + \overline{\mathbb{Q}}_\alpha (\gamma_\mu)^\alpha{}_\beta \psi^\beta + \overline{\psi}_\beta (\gamma_\mu)^\beta{}_\alpha \mathbb{Q}^\alpha + \frac{1}{2}\omega^{ab}_\mu \mathbb{J}_{ab} \,, \tag{66}$$

where ψ is charged.
We can consider $\mathbb{A} \in osp(1|2)$ as well

$$\mathbb{A}_\mu = \cancel{A_\mu \mathbb{K}} + \overline{\mathbb{Q}}_\alpha (\gamma_\mu)^\alpha{}_\beta \psi^\beta + \cancel{\overline{\psi}_\beta (\gamma_\mu)^\beta{}_\alpha \mathbb{Q}^\alpha} + \frac{1}{2}\omega^{ab}_\mu \mathbb{J}_{ab} \,, \tag{67}$$

where ψ satisfies Majorana condition.

Riemann-Cartan-Sciama-Kibble gravity,

$$\mathcal{L}_{\text{RCSK}} = \sqrt{-g}R \,, \tag{68}$$

where ω^{ab} and e^a are independent, Cartan '22 Sciama '64, Kibble '61.
Riemann-Cartan space $d = 1 + n$: $x^\mu = (x^0, \cdots, x^n)$, $\nabla g = 0$

$$e^a = e^a{}_\mu dx^\mu \,, \quad \omega^{ab} = \omega^{ab}{}_\mu dx^\mu \,, \tag{69}$$

- Independent notions: metricity (e^a) and parallelism (ω^{ab}).
- Geodesics (shortest path): $\delta S = 0$, $S = \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$, Parallel transport ('straightest' path): $\nabla V = 0$ (or $\sim V$).
- metric: kinetic terms and energy tensor, connection: couplings.
- Cartan: economy of assumptions, Einstein: economy of number of independent fields.

Reviews: Trautman 0606062, Zanelli 0502193.

Invariant gravity theories in $d = 4$

$$
\begin{aligned}
E_4 &= \epsilon_{abcd} R^{ab} R^{cd} \,, && (70) \\
\mathcal{L}_{EH} &= \epsilon_{abcd} R^{ab} e^c e^d \,, && (71) \\
\mathcal{L}_{\Lambda} &= \epsilon_{abcd} e^a e^b e^c e^d \,, && (72) \\
C_2 &= R^a{}_b R^b{}_a \,, && (73) \\
\mathcal{L}_{T_1} &= \epsilon_{abcd} R^{ab} R^{cd} \,, && (74) \\
\mathcal{L}_{T_2} &= \epsilon_{abcd} R^{ab} R^{cd} \,, && (75) \\
& && (76)
\end{aligned}
$$

Troncoso, Zanelli, Class. Quan. Grav 17 (2000) 4451.
Theories with torsion:

- Extended PPN formalism (constraints using Gravity Prove B): Mao et al Phys Rev D '07
- thorough analysis (& counter examples): Hayashi et al Phys Rev D '79
- Kleinert EJTP '10: dislocations and disclinations in a 'world crystal'.
- Richard Hammond (not the one of Top Gear): "The necessity of torsion..." Int. J. Mod. Phys. D, 19, 2413 (2010).
- SUGRAs.

Sensible invariants

Curvature:

$$\mathbb{F} = \mathcal{F}\mathbb{K} + \overline{\mathbb{Q}}^i_\alpha \mathcal{F}_i^\alpha + \mathcal{F}^a \mathbb{J}_a + \frac{1}{2}\mathcal{F}^{ab}\mathbb{J}_{ab}\,, \tag{77}$$

where

$$\mathcal{F} = F - \frac{i}{2}(\sigma^3)_i{}^j \overline{\psi}^i \not{e}\not{e}\psi_j\,, \tag{78}$$

$$\mathcal{F}_i = \hat{\nabla}(\not{e}\psi_i)\,, \tag{79}$$

$$\mathcal{F}^a = Df^a + \frac{1}{2}\overline{\psi}^i \not{e}\gamma^a \not{e}\psi_i\,, \tag{80}$$

$$\mathcal{F}^{ab} = R^{ab} + f^a f^b - \frac{1}{2}\overline{\psi}^i \not{e}\gamma^{ab}\not{e}\psi_i\,, \tag{81}$$

some shortcuts:

$$\not{e} = e^a\gamma_a\,, \quad \not{\omega} = \omega^{ab}\gamma_{ab}\,, \tag{82}$$

$$F = dA\,, \tag{83}$$

$$Df^a = df^a + \omega^a{}_b f^b\,, \tag{84}$$

$$R^{ab} = d\omega^{ab} + \omega^a{}_c\omega^{cb}\,, \tag{85}$$

and $\hat{\nabla}$ is the covariant derivative for the full $U(1)\otimes SO(3,2)$ gauge group in the $s = 1/2$ representation

$$\hat{\nabla}_i{}^j(\not{e}\psi_j) = \left[\delta_i^j d(\ ) - iA(\sigma^3)_i{}^j + \delta_i^j\left(\frac{1}{2}\not{f} + \frac{1}{4}\not{\omega}\right)\right](\not{e}\psi_j)\,, \tag{86}$$