Covariant techniques in projective and harmonic superspace

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The key people

- Harmonic superspace was largely developed in 1984.

  Galperin, Ivanov, Kalitsyn, Ogievetsky, Sokatchev

  (see monograph [GIOS])

- Projective superspace was developed also in 1984 by efforts of

  Karlhede, Lindström, Roček

  Work in later years included...

  Gates, Gonzalez-Rey, Hitchin, Kuzenko, Wiles, von Unge

  Supergravity developments based on 5D work of

  [Kuzenko, Tartaglino-Mazzucchelli '08]

Motivation: Why study these superspaces?

Matter actions with $\mathcal{N} = 2$ SUSY involve

- vector multiplets describing a special Kähler manifold
- hypermultiplets parametrizing a hyperkähler manifold (rigid SUSY) or quaternion-Kähler manifold (local SUSY)

Key distinction:

- Vector multiplets are off-shell but hypermultiplets are on-shell.

Some important ramifications of hypers being on-shell:

- Hyperkähler / QK are harder to construct than special Kähler.
- Higher-derivative actions easier for vector multiplets.
- Localization easily applied to vector multiplets (even in SUGRA) but trickier for hypermultiplets.

Harmonic and projective superspace allow off-shell hypermultiplets.

Outline

1. Review: Hypermultiplets in harmonic and projective superspace
2. Connecting projective to harmonic superspace
3. Applications: Sigma models and supergravity

Hypermultiplet superfields and off-shell representations

The free $N = 2$ (Fayet-Sohnius) hypermultiplet consists of f^i, ψ_α, $\bar{\chi}_{\dot{\alpha}}$. SUSY closes on-shell. Its superfield is given by

$$q^i(\theta, \bar{\theta}) = f^i + \theta^i \psi + \bar{\theta}_i \bar{\chi} + (x\text{-derivative terms})$$

$$D_\alpha^{(i} q^{j)} = \bar{D}_{\dot{\alpha}}^{(i} q^{j)} = 0 \quad \Longrightarrow \quad \Box q^i = 0$$

Idea of harmonic and projective superspace:

Separate the constraint into a kinematic and a dynamical piece by introducing auxiliary manifold with coordinate v^{i+}.

1. $D_\alpha^+ \mathcal{Q}^+ = \bar{D}_{\dot{\alpha}}^+ \mathcal{Q}^+ = 0$ for $D_\alpha^+ = v_i^+ D_\alpha^i$ — analyticity condition
2. $\mathcal{Q}^+ = q^i v_i^+$ — equation of motion

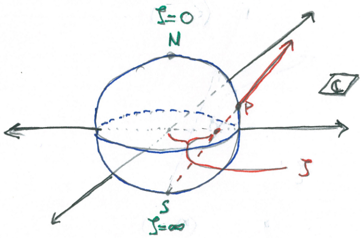
idea goes back to Rosly
(related to idea of Witten)

Harmonic coordinates on $S^2 \cong \mathbb{C}P^1$

Both harmonic and projective superspace use $S^2 \cong \mathbb{C}P^1$.

- Introduce harmonics v^{i+} and v_i^- with $v_i^- = (v^{i+})^*$ and $v^{i+} v_i^- = 1$.
- Identify $v^{i\pm} \sim e^{\pm i\psi} v^{i\pm}$.

  A useful choice: $v^{i+} \sim \dfrac{(1,\zeta)}{\sqrt{1+\zeta\bar\zeta}}$. ζ describes $\mathbb{C}P^1 \cong S^2$.



North pole
$\zeta = 0$ or $v^{i+} \sim (1, 0)$

South pole
$\zeta = \infty$ or $v^{i+} \sim (0, 1)$

Differences between harmonic and projective superspace

The differences between harmonic and projective superspace lie in the dependence on the S^2.

Harmonic superspace

Functions are globally-defined

$$\mathcal{Q}^+ = q_i v^{i+} + q_{ijk} v^{i+} v^{j+} v^{k-} + \cdots$$

Free EOM: $D^{++}\mathcal{Q}^+ = 0$ where

$$D^{++} v^{i+} = 0\ , \qquad D^{++} v_i^- = v_i^+\ ,$$

$$\implies \quad \mathcal{Q}^+ = q_i v^{i+}$$

Projective superspace

Functions are holomorphic on S^2, locally defined near N or S.
e.g. $\mathcal{Q}^+$ is holomorphic near N.
It is arctic.

$$\mathcal{Q}^+ = v^{\underline{1}+} \sum_{n=0}^{\infty} q_n \zeta^n$$

Free EOM: $\mathcal{Q}^+$ is also holomorphic near south pole (antarctic).

$$\implies \quad \mathcal{Q}^+ = q_1 v^{\underline{1}+} + q_2 v^{\underline{2}+}$$

Harmonic vs projective: Comparisons

- Harmonic superspace nicely accommodates gauge and supergravity prepotentials. (Good for quantum calculations.)
- Projective superspace has useful covariant formulations where prepotentials are hidden within covariant derivatives. (Good for derivation of covariant component actions.)

Compare with $4D$ $N = 1$ superspace: chiral multiplet with charge q

ϕ is conventionally chiral

$$\mathcal{L} = \int \mathrm{d}^4\theta \, \bar{\phi} \, e^{qV} \phi \, , \quad \bar{D}_{\dot{\alpha}}\phi = 0$$

Impose Wess-Zumino gauge on V.

$$\mathcal{L} = -\partial_m \bar{\phi} \partial^m \phi - iqA^m (\bar{\phi} \overleftrightarrow{\partial}_m \phi) - q^2 A^2 \bar{\phi}\phi + \cdots$$

Φ is covariantly chiral

$$\mathcal{L} = \int \mathrm{d}^4\theta \, \Phi^\dagger \Phi \, , \quad \bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$$

No need for any gauge choice.

$$\mathcal{L} = -\mathcal{D}_m \bar{\phi} \mathcal{D}^m \phi + \cdots$$

Supergravity case is similar but even more tricky with prepotentials!

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Connecting projective to harmonic superspace

Main issue: they involve the same space ($S^2 \cong \mathbb{C}P^1$) but different fields (globally defined vs. holomorphic)

Two alternative ways of addressing this:

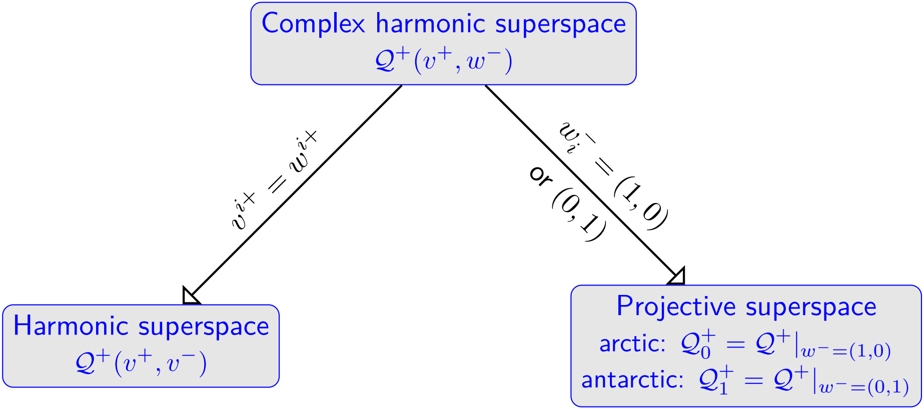
1. Deform the fields. [Kuzenko '98]
   Embed projective multiplets into globally defined harmonic multiplets – holomorphic everywhere except at the poles.
2. Deform the space. [Jain,Siegel '09; DB '12]
   Complexify the S^2 of harmonic superspace to $\mathbb{C}P^1 \times \mathbb{C}P^1$.
   Identify projective superspace S^2 with the first $\mathbb{C}P^1$.
   Second $\mathbb{C}P^1$ is additional auxiliary structure.
   We will take this approach.

The second method is similar to an approach in twistor theory. [Newman '86].

How do we complexify S^2 to $\mathbb{C}P^1 \times \mathbb{C}P^1$?

- Take harmonics u^{i+} and u_i^-, keep $u^{i+}u_i^- = 1$, but now $u_i^- \neq (u^{i+})^*$:

$$u^{i+} = v^{i+} \ , \qquad u_i^- = \frac{w_i^-}{(v^+ w^-)} \ , \qquad (v^{i\pm}, w^{i\pm}) \in \mathbb{C}P^1 \times \mathbb{C}P^1$$



Outline

Sigma models: Harmonic vs. projective [DB '12]

Harmonic solution [GIOS]

- Action
$$\mathscr{L}^{+4} = -\frac{1}{2}\mathcal{P}^+ \overleftrightarrow{D}^{++} \mathcal{Q}^+ + \mathcal{H}^{+4}$$
- Equations of motion
$$D^{++}\mathcal{Q}^+ = \frac{\partial \mathcal{H}^{+4}}{\partial \mathcal{P}^+}$$
$$D^{++}\mathcal{P}^+ = -\frac{\partial \mathcal{H}^{+4}}{\partial \mathcal{Q}^+}$$

Projective solution

[Gates, Kuzenko; Lindström, Roček]

- Action
$$\mathscr{L}^{++} = \mathcal{F}^{++}(\mathcal{Q}_0, \mathcal{P}_1)$$
- Equations of motion
$$\mathcal{P}_0^+ = \frac{\partial \mathcal{F}^{++}}{\partial \mathcal{Q}_0^+} \qquad \text{dual arctic}$$
$$\mathcal{Q}_1^+ = \frac{\partial \mathcal{F}^{++}}{\partial \mathcal{P}_1^+} \qquad \text{dual antarctic}$$

Compare to classical mechanics:

- Action principle: $F = \frac{1}{2}(q_0 p_0 + q_1 p_1) + \int_{t_0}^{t_1} \mathrm{d}t \Big(-\frac{1}{2} p \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} q + H \Big)$
- Using Hamilton's equations, $F(q_0, p_1)$ is a canonical transformation:
$$p_0 = \frac{\partial F}{\partial q_0}, \quad q_1 = \frac{\partial F}{\partial p_1}$$
- Projective actions / solutions can be derived from harmonic ones. Unifies two generating schemes for hyperkähler manifolds. [DB '12]

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Supergravity: Projective superspace (review)

- Key idea: [Kuzenko '07]

  $SU(2)_R$ of superconformal group $\equiv SU(2)$ isometry group of S^2

  this tells us that we must geometrize the R-symmetry group!

Curved projective superspace: $\mathcal{M}^{4|8} \times S^2$

[Kuzenko, Lindström, Roček, Tartaglino-Mazzucchelli '08] and [DB '14]

$$E_{\mathcal{M}}{}^{\mathcal{A}} = \begin{pmatrix} E_M{}^A & \mathcal{V}_M{}^{\underline{a}} \\ 0 & \mathcal{V}_{\underline{m}}{}^{\underline{a}} \end{pmatrix}$$

Analogous to the placement of the gravitino in the supervielbein $E_M{}^A$

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$E_M{}^A$: vielbein on $\mathcal{M}^{4|8}$

$\mathcal{V}_M{}^{\underline{a}}$: $SU(2)_R$ connection on $\mathcal{M}^{4|8}$

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vielbein on $\mathcal{M}^{4|8}$

$SU(2)_R$ connection on $\mathcal{M}^{4|8}$

vielbein, spin connection on S^2

Analogous to the placement of the gravitino in the supervielbein $E_M{}^A$

- Follow the flat case and embed curved projective superspace $(\mathcal{M}^{4|8} \times S^2)$ into curved harmonic $(\mathcal{M}^{4|8} \times \mathbb{C}P^1 \times \mathbb{C}P^1)$.
- Complication: "extra" $\mathbb{C}P^1$. Solution: attach "flat" $SU(2)$ group to sugra.

$$E_{\mathcal{M}}{}^{\mathcal{A}} = \begin{pmatrix} E_M{}^A & \mathcal{V}_M{}^{\underline{a}} & 0 \\ 0 & \mathcal{V}_{\underline{m}}{}^{\underline{a}} & 0 \\ 0 & 0 & \mathcal{W}_{\underline{\dot{m}}}{}^{\underline{\dot{a}}} \end{pmatrix}$$

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Supergravity: Harmonic superspace [DB 1408.xxxx]

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Take the framework for a test drive

- Projective and harmonic descriptions of sigma model coupled to conformal supergravity

$$S = -\frac{1}{2\pi} \oint_{\mathcal{C}} v_i^+ \mathrm{d}v^{i+} \int \mathrm{d}^4x\, \mathrm{d}^4\theta^+\, \mathcal{E}\mathcal{F}^{++} ~,$$

$$S = \frac{i}{2\pi} \oint_{\mathcal{S}} v_i^+ \mathrm{d}v^{i+} \wedge w_j^- \mathrm{d}w^{j-} \int \mathrm{d}^4x\, \mathrm{d}^4\theta^+\, \mathcal{E} \Big(-\frac{1}{2} \mathcal{P}^+ \overleftrightarrow{D}^{(0,2)} \mathcal{Q}^+ + \mathcal{H}^{(2,2)} \Big)$$

- Component reduction gives sigma model (a hyperkähler cone) coupled to conformal supergravity.
- Results agree with each other and with known component results of [de Wit, Kleijn, Vandoren '99].

Conclusion and open questions

- Harmonic and projective superspaces are not intrinsically different formulations of off-shell $\mathcal{N} = 2$ superspace but are rather complementary.
- Understanding projective superspace tells us how to introduce covariant formulation of harmonic superspace coupled to conformal supergravity.
- Covariant formulation readily admits higher-derivative actions. Can we construct these with hypermultiplets using either projective superspace or harmonic superspace? *see e.g.* [DB, Kuzenko '10]
- Can we learn (more) about prepotentials in projective superspace using harmonic? Advances in understanding gauge prepotential already due to [Jain,Siegel]. What about supergravity?

stay tuned...