# LOCALIZATION on FOUR-MANIFOLDS, CASIMIR ENERGY and GRAVITY DUALS

**Davide Cassani**

**King’s College London**

SUSY 2014 Manchester

# Setting

*As all beautiful things,*

exact results in QFT are highly desirable but rare

- **supersymmetry** has proven a very successful theoretical tool

  e.g. non-renormalization theorems, moduli space of vacua, …

- In the last few years: **many new exact results**

  based on the technique of **supersymmetric localization**

- In this talk, we will see a particularly interesting example

# Outline

❶ supersymmetry in curved space and localization

❷ partition function on Hopf surfaces

❸ comparison with the index and Casimir energy

❹ gravity duals

**Based on** **1402.2278 with D. Martelli**

**1405.5144 with B. Assel and D. Martelli**

# QFT path integral

$\int \mathcal{D}\Phi \, \mathcal{O} \, e^{iS[\Phi]}$ ➜ gives a non-perturbative definition of a QFT

- hard to compute: ⚠ infinite-dimensional $\mathcal{D}\Phi(x)$
  ⚠ integrand oscillates $e^{iS[\Phi]}$
  ⚠ both IR and UV divergent
- it becomes more tractable :
  - ✤ in Euclidean signature $e^{iS[\Phi]} \longrightarrow e^{-S[\Phi]}$
  - ✤ on compact manifolds : finite radius acts as an IR regulator
- dramatic simplification in **supersymmetric** QFT
  - ✤ improved UV behavior
  - ✤ **localization :** infinite-dimensional path integral
    reduces to a finite-dimensional one

# Localization

- With some assumptions,

  the supersymmetric path integral can be deformed so that

  - it is dominated by simple supersymmetric configurations $\mathbf{\Phi_0}$
  - saddle point approximation becomes **exact**

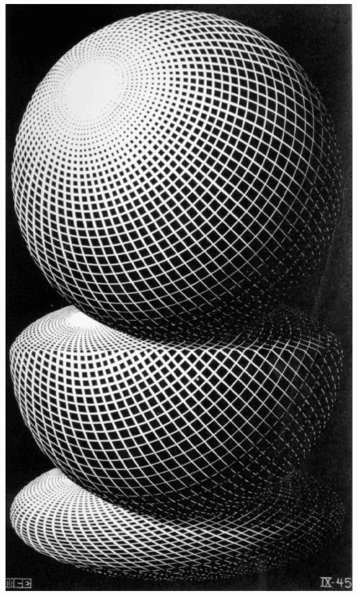
➜ **huge simplification !**

$$Z = \int \mathcal{D}\Phi_0 \, e^{-S[\Phi_0]} \frac{1}{\mathrm{Sdet}[\text{kinetic operator for } \delta\Phi]}$$

often $\Phi_0 = \text{const}$, so $\mathcal{D}\Phi_0 \to d\Phi_0$

- In the last years the exact partition function has been computed for many theories on various geometries, in different dimensions.

  Many applications.



# Partition function with sources

Need to place our field theory on a Riemannian manifold by preserving susy

- Couple it to background fields :

$$S[\Phi; \underset{\substack{\nearrow\\ \text{background}\\ \text{gauge field}}}{A_\mu}, \underset{\substack{\nwarrow\\ \text{background}\\ \text{curved metric}}}{g_{\mu\nu}}] \; = \; S_0[\Phi] + \int \underbrace{\left(A_\mu \underset{\substack{\nearrow\\ \text{conserved}\\ \text{current}}}{j^\mu} + g^{\mu\nu} \underset{\substack{\nwarrow\\ \text{energy-momentum}\\ \text{tensor}}}{T_{\mu\nu}} + \ldots\right)}_{\text{supermultiplet}}$$

- Partition function :

$$Z[A_\mu, g_{\mu\nu}] \; = \; \int \mathcal{D}\Phi \, e^{-S[\Phi; A_\mu, g_{\mu\nu}]}$$

- can compute a specific set of correlators :

$$-\frac{\delta}{\delta A_\mu} \log Z[A] \; = \; \langle j^\mu \rangle \qquad\qquad -\frac{\delta}{\delta g^{\mu\nu}} \log Z[g] \; = \; \langle T_{\mu\nu} \rangle$$

# Supersymmetric backgrounds

- Which curved backgrounds preserve supersymmetry?

For a four-dimensional N=1 theory with an R-symmetry

Klare, Tomasiello, Zaffaroni; Dumitrescu, Festuccia, Seiberg '12

- **one** supercharge $\Longleftrightarrow$ complex manifold with Hermitian metric
- **two** supercharges (of opposite R-charge) ➜ complex isometry $K$

  other background fields, including $A_\mu$ coupling to R-current, fixed by supersymmetry

Focus on second case : localization more powerful

# Hopf surfaces

- Choose $S^1 \times S^3$ topology.

Assel, D.C., Martelli
see also Closset, Shamir

# Hopf surfaces

Assel, D.C., Martelli

see also Closset, Shamir

- Choose $S^1 \times S^3$ topology.
- Complex manifolds with $S^1 \times S^3$ topology are **Hopf surfaces** $\mathcal{H}_{p,q}$

  defined as a quotient of $\mathbb{C}^2 - (0,0)$ $\qquad (z_1, z_2) \sim (pz_1, qz_2)$

  $p = \mathrm{e}^{-2\pi b_1}\,, \quad q = \mathrm{e}^{-2\pi b_2}$ : complex structure moduli

# Hopf surfaces

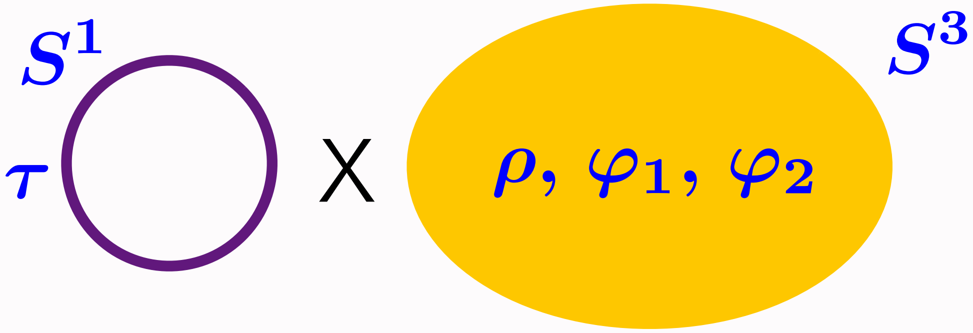
Assel, D.C., Martelli

see also Closset, Shamir

- Choose $S^1 \times S^3$ topology.
- Complex manifolds with $S^1 \times S^3$ topology are **Hopf surfaces** $\mathcal{H}_{p,q}$

defined as a quotient of $\mathbb{C}^2 - (0,0)$ $\quad (z_1, z_2) \sim (pz_1, qz_2)$

$p = \mathrm{e}^{-2\pi b_1}, \quad q = \mathrm{e}^{-2\pi b_2}$ : complex structure moduli



# Hopf surfaces

Assel, D.C., Martelli

see also Closset, Shamir

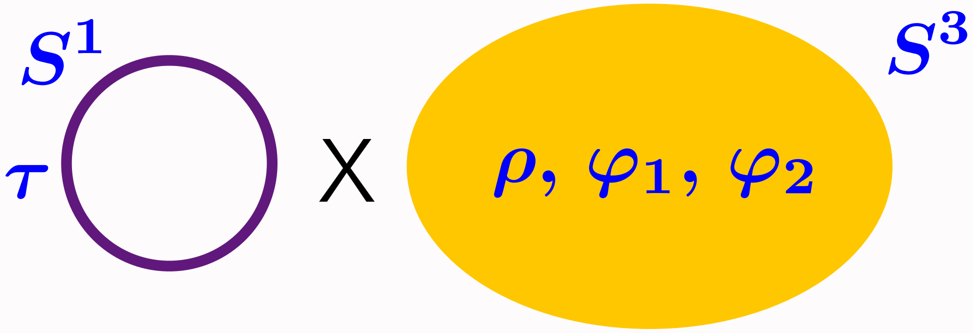
- Choose $S^1 \times S^3$ topology.
- Complex manifolds with $S^1 \times S^3$ topology are **Hopf surfaces** $\mathcal{H}_{p,q}$

  defined as a quotient of $\mathbb{C}^2 - (0,0)$ $\quad (z_1, z_2) \sim (pz_1, qz_2)$

  $p = \mathrm{e}^{-2\pi b_1} , \quad q = \mathrm{e}^{-2\pi b_2}$ : complex structure moduli

- two supercharges: complex Killing vector

$$K = b_1 \frac{\partial}{\partial \varphi_1} + b_2 \frac{\partial}{\partial \varphi_2} - i \frac{\partial}{\partial \tau}$$



$S^3$ as torus fibration over an interval, $b_1$ , $b_2$ $\in \mathbb{R}$ for simplicity

# Hopf surfaces

Assel, D.C., Martelli

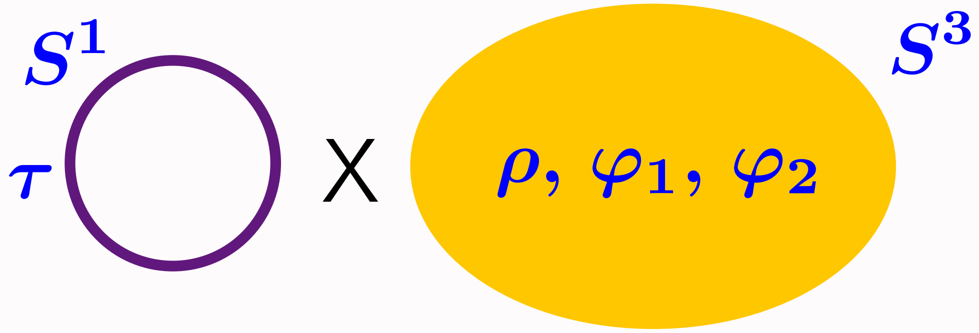
see also Closset, Shamir

- Choose $S^1 \times S^3$ topology.
- Complex manifolds with $S^1 \times S^3$ topology are **Hopf surfaces** $\mathcal{H}_{p,q}$

  defined as a quotient of $\mathbb{C}^2 - (0,0)$ $\quad (z_1, z_2) \sim (pz_1, qz_2)$

  $p = \mathrm{e}^{-2\pi b_1}, \quad q = \mathrm{e}^{-2\pi b_2}$ : complex structure moduli

- two supercharges: complex Killing vector



$$K = b_1 \frac{\partial}{\partial \varphi_1} + b_2 \frac{\partial}{\partial \varphi_2} - i \frac{\partial}{\partial \tau}$$

$S^3$ as torus fibration over an interval, $b_1$ , $b_2 \in \mathbb{R}$ for simplicity

- compatible metric still very general :

$$\mathrm{d}s^2 = \Omega^2(\rho)\mathrm{d}\tau^2 + f^2(\rho)\mathrm{d}\rho^2 + m_{IJ}(\rho)\mathrm{d}\varphi_I \mathrm{d}\varphi_J \qquad I, J = 1, 2$$

# Localization on Hopf surfaces

- Consider partition function of an Euclidean theory on $\mathcal{H}_{p,q}$ , with
  - vector multiplet for general gauge group
  - (charged) chiral multiplets, with superpotential

# Localization on Hopf surfaces

- Consider partition function of an Euclidean theory on $\mathcal{H}_{p,q}$ , with
  - vector multiplet for general gauge group
  - (charged) chiral multiplets, with superpotential

- Adding a suitable susy-exact deformation term,

path integral localizes $\left\{ \begin{array}{l} \text{dynamical gauge field } \mathcal{A}_\tau = \mathbf{const} \\ \text{all other fields vanishing} \end{array} \right.$

# Localization on Hopf surfaces

- Consider partition function of an Euclidean theory on $\mathcal{H}_{p,q}$ , with
  - vector multiplet for general gauge group
  - (charged) chiral multiplets, with superpotential

- Adding a suitable susy-exact deformation term,

  path integral localizes $\left\{ \begin{array}{l} \text{dynamical gauge field } \mathcal{A}_\tau = \mathbf{const} \\ \text{all other fields vanishing} \end{array} \right.$

- Integral over all field fluctuations around this localization locus :

$$\mathbf{Sdet}[\textbf{kinetic operator for } \delta\Phi]$$

Compute it building on 3d results

Alday, Martelli, Richmond, Sparks

- susy ➜ many cancellations between eigenvalues
- left with ∞ product over 3 integers (from Fourier modes on $U(1)^3$ )
- regularized using triple Gamma and zeta functions

# Localization on Hopf surfaces

Result :

$$Z[\mathcal{H}_{p,q}] = \underbrace{e^{-\mathcal{F}(p,q)}}_{\text{prefactor}} \underbrace{\frac{(p;p)^{r_G}(q;q)^{r_G}}{|\mathcal{W}|} \int_{T^{r_G}} \frac{dz}{2\pi i z} \prod_{\alpha\in\Delta_+} \theta\left(z^{\alpha},p\right)\theta\left(z^{-\alpha},q\right) \prod_J \prod_{\rho\in\Delta_J} \Gamma_e\left(z^{\rho}(pq)^{\frac{r_J}{2}},p,q\right)}_{\text{supersymmetric index } \mathcal{I}(p\,,q)}$$

Witten index $\mathrm{tr}(-1)^F$, refined by fugacities $p\,,\;q$

counts certain BPS states

- General arguments show that $Z$ is a holomorphic function of the complex structure parameters and does not depend on Hermitian metric

Closset, Dumitrescu, Festuccia, Komargodski

- $Z[\mathcal{H}_{p,q}]$ conjectured to compute $\mathcal{I}(p\,,q)$

➜ we have explicitly checked this. Found an **extra contribution** $\mathcal{F}(p,q)$

# Localization on Hopf surfaces

$$Z[\mathcal{H}_{p,q}] = \mathrm{e}^{-\mathcal{F}(p,q)} \mathcal{I}(p\,,q)$$

index

$$\boxed{\mathcal{F}(p,q) = \frac{4\pi}{3}\left(b_1 + b_2 - \frac{b_1 + b_2}{b_1 b_2}\right)(a - c) + \frac{4\pi}{27}\frac{(b_1 + b_2)^3}{b_1 b_2}(3\,c - 2\,a)}$$

$$p = \mathrm{e}^{-2\pi b_1}\,, \quad q = \mathrm{e}^{-2\pi b_2}$$

$$a = \frac{3}{32}\left(3\,\mathrm{tr}R^3 - \mathrm{tr}R\right), \qquad c = \frac{1}{32}\left(9\,\mathrm{tr}R^3 - 5\,\mathrm{tr}R\right)$$

$R$ : fermionic R-charge

SCFT central charges

- appears to be physical (non-removable by supersymmetric local 4d counterterm)
- limit of large $S^1$ ➜ yields a **supersymmetric Casimir energy**
- related to anomalies?
- dominates $Z$ at large N ➜ **prediction for dual supergravity solutions**

# Gravity duals

AdS/CFT master equation (at large N)

$$\mathrm{e}^{-S_{\mathrm{gravity}}[M_5]} = Z[M_4]$$

$M_4 = \partial M_5$ QFT background fields ⟺ gravity boundary conditions

➜ holographic evaluation of the QFT partition function

➜ can make highly non-trivial tests of AdS/CFT and gain useful insight for field theory computations

- When $\partial M_5 = \mathcal{H}_{p,q}$ , our prediction from localization (at large N) :

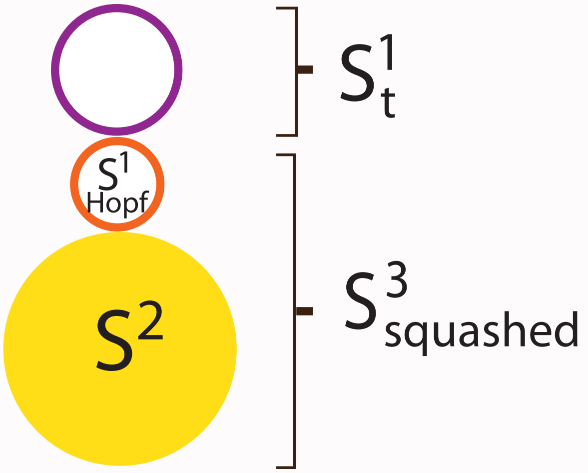
$$S_{\mathrm{5d\,sugra}}[M_5] = \frac{\pi^2}{54 G_5} \frac{(b_1 + b_2)^3}{b_1 b_2}$$

# New supergravity solution

D.C., Martelli '14

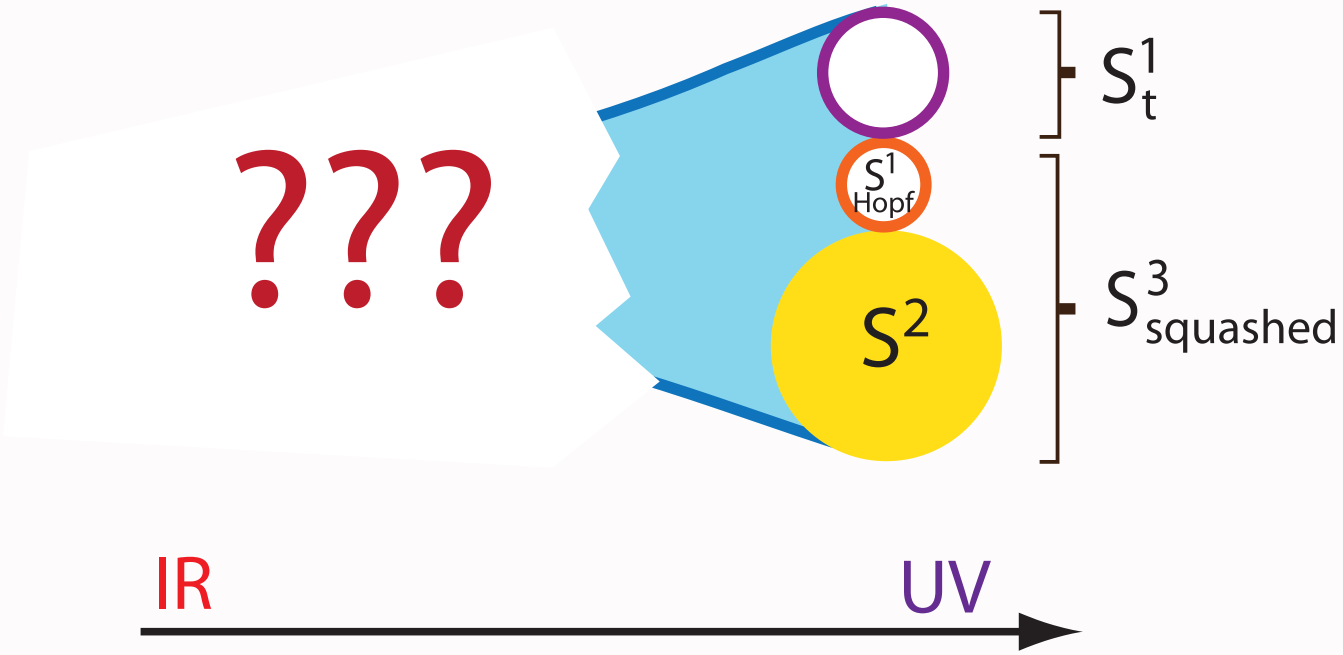
We took a first step :

- considered $S^1 \times S^3_{\text{squashed}}$



more symmetry U(1)xU(1)xU(1) ➜ SU(2)xU(1)xU(1)

- studied 5d supergravity susy equations with these boundary conditions

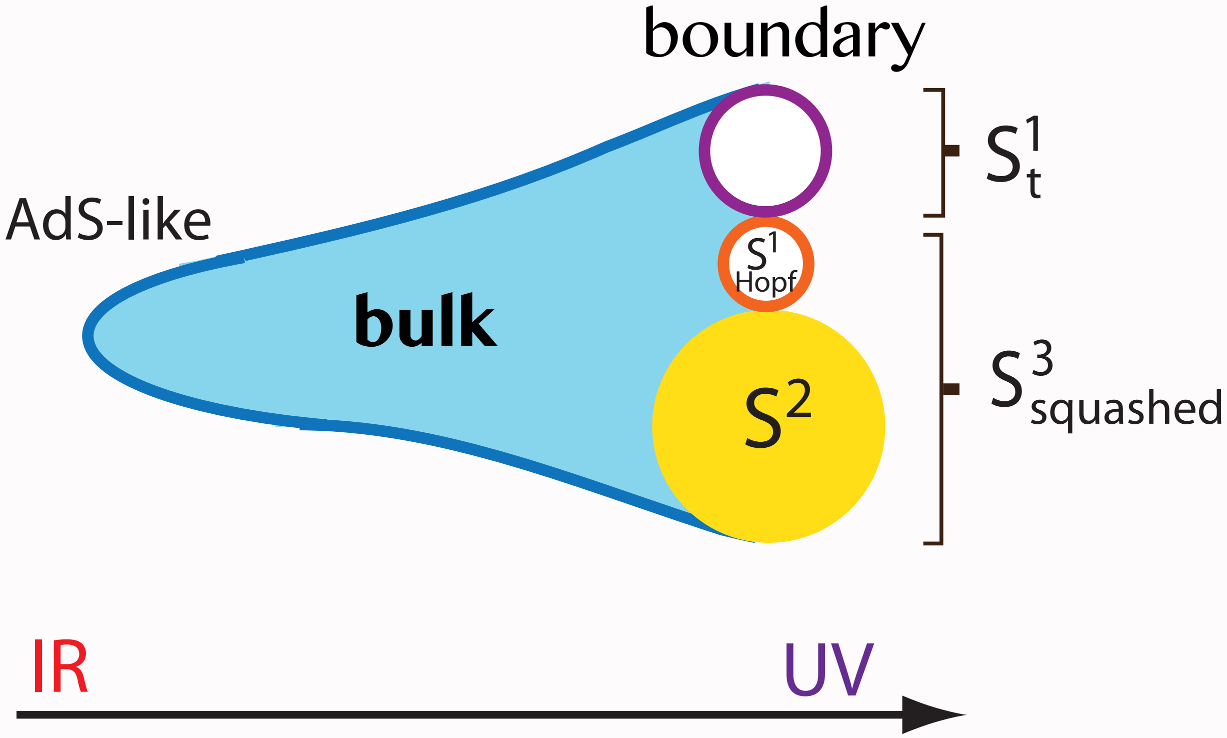


# New supergravity solution

D.C., Martelli '14

**found a new one-parameter family of solutions**

- regular
- no horizon
- family parameterized by squashing of $S^3$



- $S_{\text{gravity}} \sim \dfrac{r_{S^1}}{r_{S^1_{\text{Hopf}}}} + \dots$ ← counterterms

agrees with field theory formula with $b_1 = b_2$

⚠ need better understanding of supersymmetric holographic renormalization

# Conclusions

- I presented an explicit computation of the partition function of N=1 gauge theories on a Hopf surface $\mathcal{H}_{p,q}$ , allowing for a general metric

Find: $Z[\mathcal{H}_{p,q}] \; = \; \mathrm{e}^{-\mathcal{F}(p,q)}\mathcal{I}(p\,,q)$

- First holographic check by constructing a new supergravity solution

- $\mathcal{F}(p,q)$ involves supersymmetric Casimir energy

  It would be interesting :

  - ◆ to explore more its meaning in field theory
  - ◆ to retrieve it in full generality in a holographic setup

    ➜ refine our understanding of gauge/gravity correspondence

... thank you for your attention

# Extra slides

# localization

fields $\Phi$ , (Euclidean) action $S[\Phi]$

$$Z = \int \mathcal{D}\Phi\, e^{-S[\Phi]} = ?$$

supersymmetry $Q$ such that $Q\,S = 0$

introduce a fermionic term $V[\Phi]$ : $Q\,(QV) = 0$

deformed partition function $Z(t) = \int \mathcal{D}\Phi\, e^{-S[\Phi]-\underset{\nwarrow \text{parameter}}{t}\,QV[\Phi]}$

independent of $t$ :

$$\frac{d}{dt}Z(t) = -\int \mathcal{D}\Phi\, QV\, e^{-S-tQV} = -\int \mathcal{D}\Phi\, Q\Big(V\, e^{-S-tQV}\Big) = 0$$

$$\rightarrow \quad \underline{Z(t=0) = Z(t\to\infty)}$$

if $QV|_{\text{bosonic}} \geq 0$, then dominant contribution at large $t$ is from $QV|_{\text{bos}} = 0$

$t \to \infty$ : saddle point approximation becomes **exact**

# localization

typical choice : $V = \int d^4x\sqrt{g}\,\left[(Q\psi)^\dagger\psi + \psi^\dagger(Q\psi^\dagger)^\dagger\right]$ $\psi$ fermions

$$QV|_{\mathbf{bosonic}} = \int d^4x\sqrt{g}\,\left(|Q\psi|^2 + |Q\psi^\dagger|^2\right) \geq 0$$

saddle points ⇔ supersymmetric configurations $Q\psi = 0 \quad Q\psi^\dagger = 0$ ★

$$\psi = \psi^\dagger = 0$$

$$\Phi = \underset{\substack{\uparrow \\ \text{solving } \bigstar}}{\Phi_0} + \frac{1}{\sqrt{t}}\,\underset{\substack{\uparrow \\ \text{fluctuation}}}{\delta\Phi} + \cancel{\ldots}$$

$$\boxed{Z = \int \mathcal{D}\Phi_0\, e^{-S[\Phi_0]}\,\frac{1}{\mathrm{Sdet}[\text{kinetic operator for } \delta\Phi]}}$$

often $\Phi_0 = \mathrm{const}$, so $\mathcal{D}\Phi_0 \to d\Phi_0$

➜ infinite-dimensional integral localizes to a finite one

# minimal d=5 gauged supergravity

- the simplest d=5 sugra with AdS vacuum

- field content $g_{\mu\nu}\,,\; A_\mu\,,\; \psi_\mu$

- bosonic lagrangian:

$$\mathcal{L}_{\text{sugra}} = \left(R + 12 - \tfrac{1}{3}F^2\right) * 1 - \tfrac{8}{27}\, A \wedge F \wedge F$$

- susy condition $\delta\psi_\mu = 0$ :

$$\left[\nabla^A_\mu - \tfrac{1}{2}\gamma_\mu - \tfrac{i}{12}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho\right)F_{\nu\rho}\right]\epsilon = 0$$

$$\nabla^A := \nabla - iA$$

- any solution lifts to type IIB supergravity on a Sasaki-Einstein$_5$