Determination of $\alpha_s(M_\tau^2)$: a conformal mapping approach

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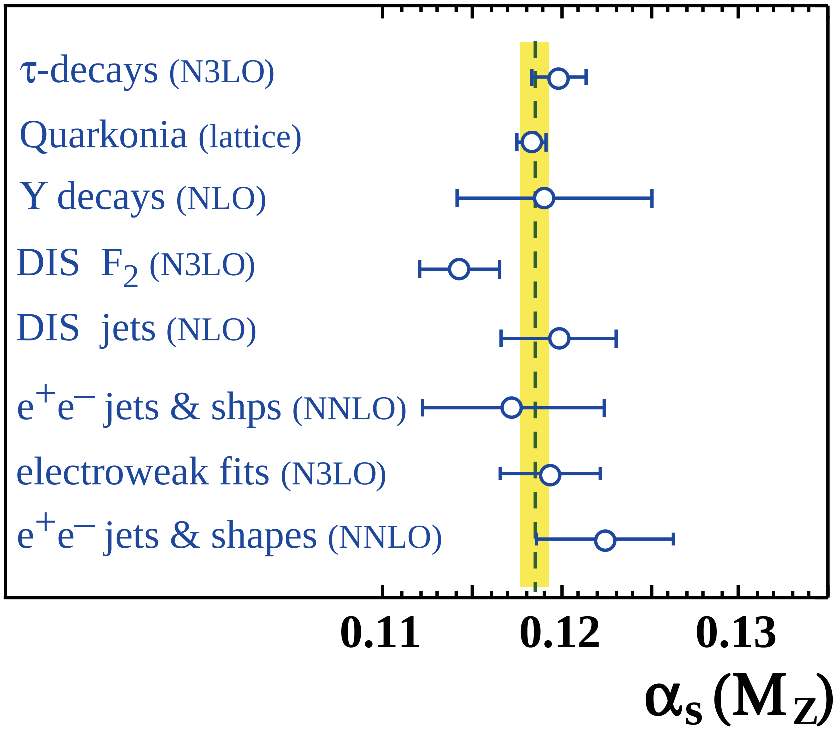
Outline

- Status of α_s determination
- α_s from τ decays: basic formulas and open problems
- Divergent series, analyticity, conformal mappings
- New expansions in perturbative QCD
- Determination of $\alpha_s(M_\tau^2)$

Status of α_s determination



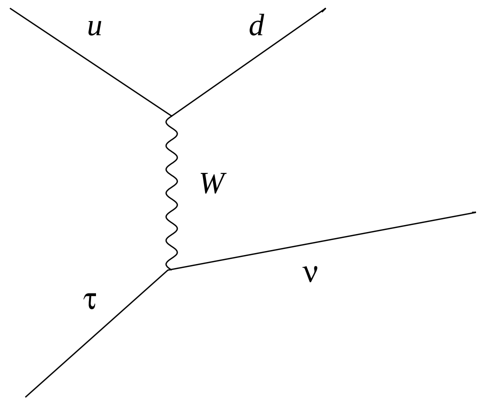
Summary of α_s measurements S. Bethke, EPJC 2009

World average 2009: $\alpha_s(M_Z^2) = 0.1184 \pm 0.0007$

Determination of α_s from τ decays

- determination of α_s at a low scale $\quad (M_\tau = 1.78\,\mathrm{GeV})$

- $\tau \longrightarrow \text{hadrons} \; + \nu_\tau$



- measured quantity LEP, CLEO

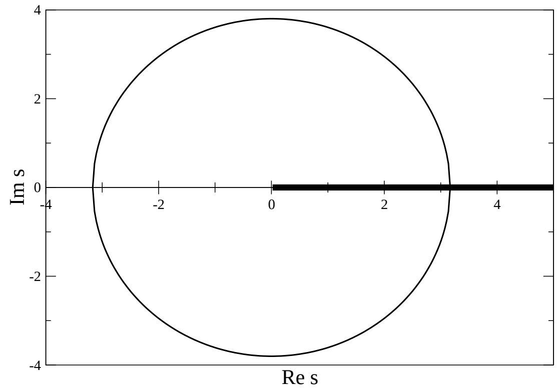
$$R_\tau = \frac{\Gamma[\tau \to \text{hadrons} \; + \nu_\tau]}{\Gamma[\tau \to \mu \; + \bar{\nu}_\mu \; + \nu_\tau]} = 3.640 \pm 0.010$$

Basic formulae

- theoretical expression

$$R_\tau = 12\pi \int_{4M_\pi^2}^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) \mathrm{Im}\,\Pi(s) \sim 1 + \delta^{(0)}$$

- causality and unitarity: $\Pi(s)$ real analytic in the s-plane cut for $s > 4M_\pi^2$



- Cauchy theorem: $\Rightarrow$

$$\boxed{\delta^{(0)} = \frac{1}{2\pi i} \oint_{|s|=M_\tau^2} \frac{ds}{s}\, \omega(s)\, \widehat{D}(s)}$$

  - $\omega(s) = 1 - 2s/M_\tau^2 + 2(s/M_\tau^2)^3 - (s/M_\tau^2)^4$
  - $\widehat{D}(s) = -\,s\,\frac{d}{ds}\left[\Pi(s)\right] - 1$ Adler function

Basic formulae

- $\alpha_s(M_\tau^2)$ determined from the equation $\boxed{\delta^{(0)}_{\rm theor} = \delta^{(0)}_{\rm exp}}$

  - $\delta^{(0)}_{\rm theor}$ calculated from the perturbation expansion

$$\widehat{D}(s) = \sum_{n\geq 1} \left[K_n + \kappa_n(s/\mu^2)\right] \left(a_s(\mu^2)\right)^n, \qquad a_s = \frac{\alpha_s}{\pi}$$

$$\kappa_n(s/\mu^2) = \sum_{k=1}^{n} \gamma_{kn} \ln^k(-s/\mu^2)$$

  - $K_1 = 1,\ K_2 = 1.64,\ K_3 = 6.37,\ K_4 = 49.08,\ K_5 \sim 283$

  - γ_{kn}: calculated in terms of K_n and the coefficients of the β function

Theoretical problems

- validity of the OPE near the timelike axis
- nonperturbative contributions (condensates)
- ambiguity of the perturbative expansion
  - choice of μ^2 (CIPT versus FOPT)
  - alternative expansions from various motivations
- present work: include theoretical knowledge about the high order behaviour of the series and the singularities at $\alpha_s = 0$

Perturbation theory: high-order behaviour

- from particular classes of Feynman diagrams $\quad K_n \sim n!$

  $\Rightarrow$ the renormalized perturbation series is divergent

- from independent arguments it is known that $\widehat{D}$, regarded as a function of α_s, is singular at $\alpha_s = 0$, at least along the whole negative real semiaxis of the α_s plane 't Hooft 1979

  - for QED these facts are known since 1952, but do not affect the precision since α is small Dyson 1952

- for a large coupling like $\alpha_s(M_\tau^2)$ these facts do matter

Need for a new expansion

- the expanded function is badly singular at the expansion point, while the powers of α_s are all holomorphic, without any singularities

- it would be more reasonable to expand $\widehat{D}$ in some special set of functions $W_n(\alpha_s)$, which resemble as much as possible the expanded Adler function in the following aspects:

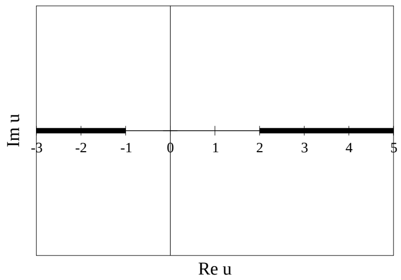
  (1) the location of singularities in the complex plane, and

  (2) the nature of the singularities

- the problem (1) can be solved rigorously in the Borel u-plane for the Borel transform $B(u)$

Borel transform of the Adler function

$$b_n = \frac{K_{n+1}}{\beta_0^n \, n!}\,, \quad n \geq 0, \quad \Rightarrow \quad B(u) = \sum_{n=0}^{\infty} b_n u^n \text{ converges in a disk}$$



$B(u)$ has singularities on the real axis for $u \leq -1$ and $u \geq 2$

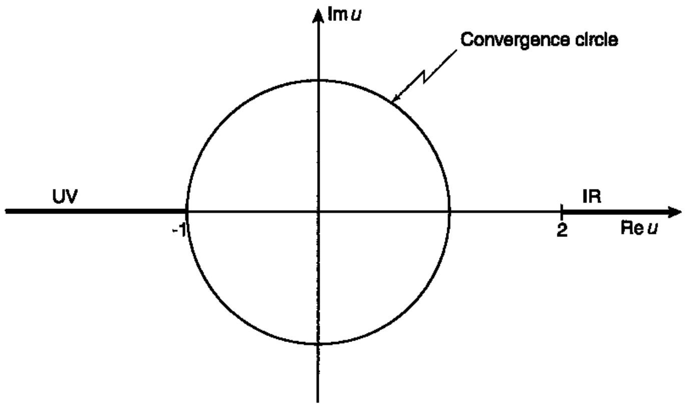
(UV and IR renormalons)

$\widehat{D}$ recovered by the prescription: $\quad \widehat{D}(s) = \frac{1}{\beta_0} \, \mathrm{PV} \int_0^{\infty} e^{-u/(\beta_0 a_s(s))} \, B(u) \, du$

- reproduces the standard expansion in powers of a_s
- consistent with general properties of $\widehat{D}(s)$ IC & Neubert, JHEP 1999

"Series acceleration" in the Borel plane

- $B(u) = \sum_{n=0}^{\infty} b_n u^n$ converges only in the disk $|u| < 1$
  - range in Laplace integral extends beyond the convergence radius



- Look for a new expansion $B(u) = \sum_{n=0}^{\infty} c_n (w(u))^n$ that:
  - converges in a larger domain
  - has a better convergence at every point

Conformal mappings

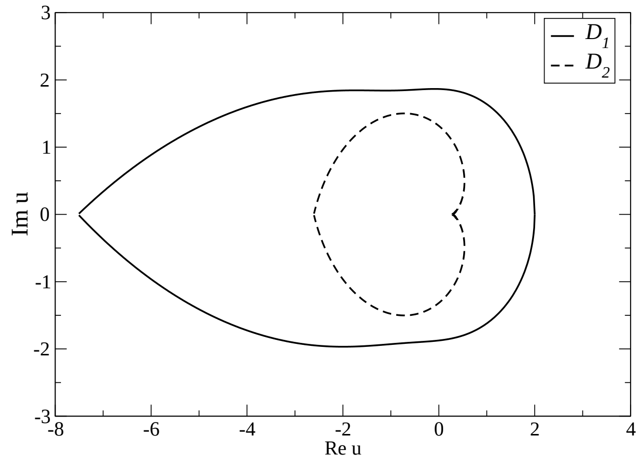
Lemma (Ciulli & Fischer, NP1961): Let D_1 and D_2 two domains in the u-plane, with D_2 included in D_1. Consider the conformal mappings

$$z_1 = \tilde{z}_1(u) : D_1 \to D = \{z_1 : |z_1| < 1\}$$

$$z_2 = \tilde{z}_2(u) : D_2 \to D = \{z_2 : |z_2| < 1\}$$

such that $z_1(0) = 0$ and $z_2(0) = 0$.

Then $|z_1(u)| < |z_2(u)|$ for all $u \in D_2$.



Proof: based on Schwarz Lemma

Schwarz Lemma: Let $D = \{z : |z| < 1\}$ be the open disk in the complex plane, centered at the origin, and $f : D \to D$ a holomorphic map such that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in D$. Moreover, if $|f(z)| = |z|$ at some $z \neq 0$, then $f(z) = e^{i\phi} z$.

Conformal mappings and series convergence

Theorem (Ciulli & Fischer, NP1961): Let $B(u)$ holomorphic in D_1 and the expansions $B(u) = \sum_0^\infty c_{n,1}(\tilde{z}_1(u))^n$ and $B(u) = \sum_0^\infty c_{n,2}(\tilde{z}_2(u))^n$ convergent in the unit disks $|z_1| < 1$ and $|z_2| < 1$, respectively. Then the first series has a better asymptotic convergence rate than the second.

Proof: We must estimate the ratio $\mathcal{R}_n = \frac{|c_{n,1}(\tilde{z}_1(u))^n|}{|c_{n,2}(\tilde{z}_2(u))^n|}$ at large n.

- $\lim_{n\to\infty}(c_{n,j})^{1/n} = 1 \;\Rightarrow |c_{n,j}| \sim e^{g_j(n)}$ where $g_j(n)/n \to 0$ for $n \to \infty$.
- $\boxed{\mathcal{R}_n \sim e^{g(n)} \cdot \rho^n}$ where $g(n) = g_1(n) - g_2(n), \;\; \rho = |z_1(u)/z_2(u)|$
- $g(n)/n \to 0$ and $\rho < 1$ cf. Lemma

$\Rightarrow \quad \ln \mathcal{R}_n \sim n[g(n)/n + \ln \rho]$ is negative for n large.

$\Rightarrow \quad \boxed{\mathcal{R}_n < 1 \text{ for large n}}$

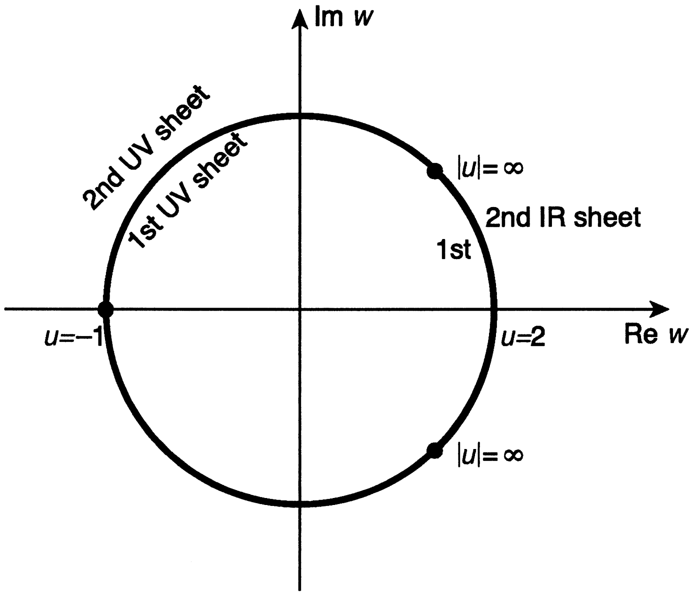
Consequences

- the "optimal" variable maps the whole holomorphy domain onto the unit disk

- For the Borel transform of Alder function

$$\boxed{\tilde{w}(u) = \frac{\sqrt{1+u} - \sqrt{1-u/2}}{\sqrt{1+u} + \sqrt{1-u/2}}}$$

IC & Fischer 1999



- maps the cut u-plane onto $|w| < 1$ in the plane $w = \tilde{w}(u)$, with $\tilde{w}(0) = 0$

- the series $B(u) = \sum_n c_n w^n, \qquad w = \tilde{w}(u)$
  - converges in the whole u-plane up to the cuts
  - best asymptotic convergence rate for interior points

"Singularity softening"

- near the first branch-points $B(u) \sim (1+u)^{-\gamma_1}$, $B(u) \sim (1-u/2)^{-\gamma_2}$, where $\gamma_1 > 0$ and $\gamma_2 > 0$ are known from RGE   Mueller 1985, Beneke et al 1997
- the convergence of the expansion can be improved by "softening" the singularities with suitable factors   Soper, Surguladze 1996
- unlike the optimal conformal mapping, "singularity softening" is not unique
- suitable choice: IC & Fischer, EPJC 2009

$$(1-w)^{2\gamma_1}(1+w)^{2\gamma_2}B(\tilde{u}(w)) = \sum_n c_n w^n$$

$\tilde{u}(w) = \frac{8w}{3-2w+3w^2}$ the inverse of $w = \tilde{w}(u)$

New perturbation expansion

- CIPT version

$$\widehat{D}(s) = \sum_n c_n \mathcal{W}_n(s)$$

$$\mathcal{W}_n(s) = \frac{1}{\beta_0} \mathrm{PV} \int_0^\infty e^{-u/(\beta_0 a_s(s))} \frac{w^n}{(1+w)^{2\gamma_1}(1-w)^{2\gamma_2}} \, du, \quad w = \tilde{w}(u)$$

- FOPT version

$$\widehat{D}(s) = \sum_n \tilde{c}_n(s) \tilde{\mathcal{W}}_n$$

$$\tilde{\mathcal{W}}_n = \frac{1}{\beta_0} \mathrm{PV} \int_0^\infty e^{-u/(\beta_0 a_s(M_\tau^2))} \frac{w^n}{(1+w)^{2\gamma_1}(1-w)^{2\gamma_2}} \, du$$

IC & Fischer, EPJC 2009

Properties of the new expansion

- when reexpanded in powers of α_s, it reproduces the coefficients K_n known from Feynman diagrams
- the functions $\mathcal{W}_n$ are singular at $\alpha_s = 0$ and their expansion in powers of the coupling are divergent
- under certain conditions, the expansion

$$\widehat{D}(s) = \sum_n c_n \mathcal{W}_n(s)$$

is convergent in a domain of the complex plane

IC & Fischer, PRD 2000, EPJC 2002

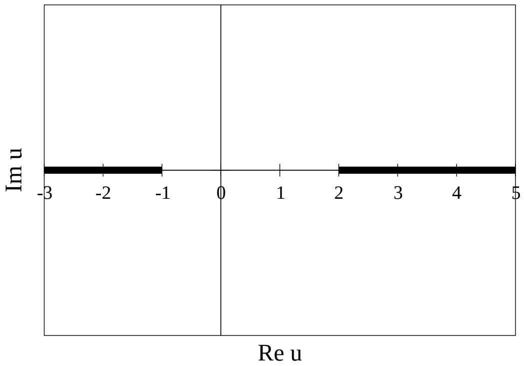
Other conformal mappings

- after "softening" the first singularities are milder (ex: $(1 - u/2)^{\gamma_2}$ with $\gamma_2 > 0$ is a branch point, but is finite at u=2)
- the effect of a mild singularity is expected to appear only at larger orders in an expansion
- at low orders we can use other conformal mappings, which account only for the next singularities in the u-plane
- define the mappings:

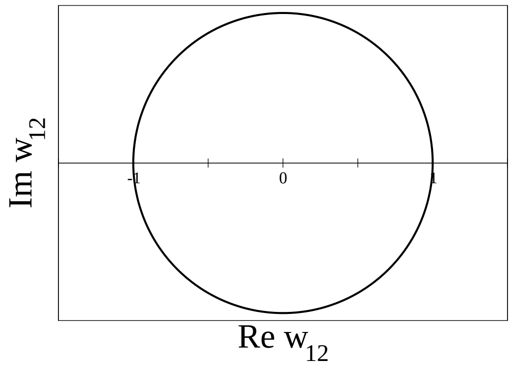
$$\tilde{w}_{jk}(u) = \frac{\sqrt{1+u/j} - \sqrt{1-u/k}}{\sqrt{1+u/j} + \sqrt{1-u/k}}, \qquad j \geq 1, \quad k \geq 2$$

$\tilde{w}_{jk}$ maps the u-plane cut for $u < -j$ and $u > k$ onto $|w_{jk}| < 1$

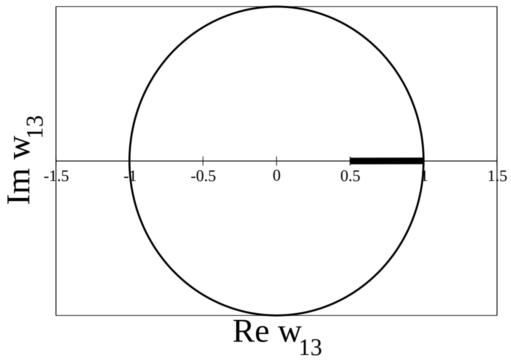
Examples



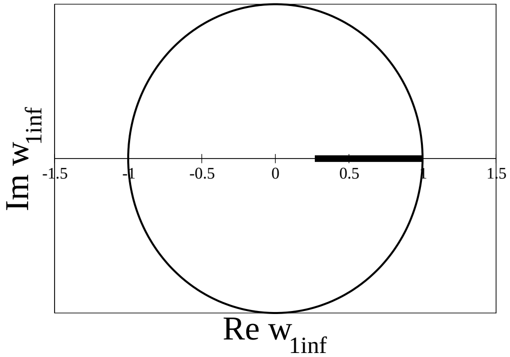
u-plane



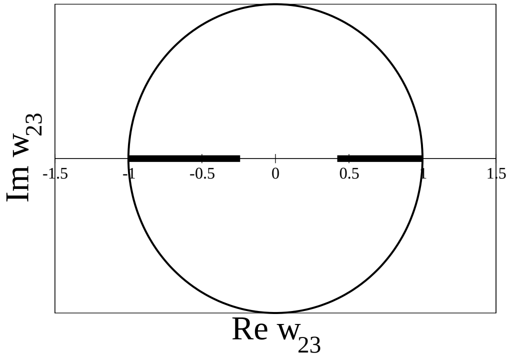
$\tilde{w}_{12}(u)$ (IC& Fischer, 1999, 2009)



$\tilde{w}_{13}(u)$ (Cvetic, Lee, 2001)



$\tilde{w}_{1\infty}(u)$ (Mueller, 1992)



$\tilde{w}_{23}(u)$

New expansions (CIPT version)

$$\widehat{D}(s) = \sum_n c_n^{jk}\, \mathcal{W}_n^{jk}(s)$$

$$\boxed{\mathcal{W}_n^{jk}(s) = \frac{1}{\beta_0} PV \int_0^\infty e^{-u/(\beta_0 a_s(s))} \frac{(\tilde{w}_{jk}(u))^n}{S_{jk}(u)}\, du}$$

- $S_{jk}(u) = \left(1 - \frac{\tilde{w}_{jk}(u)}{\tilde{w}_{jk}(-1)}\right)^{\gamma_1'} \left(1 - \frac{\tilde{w}_{jk}(u)}{\tilde{w}_{jk}(2)}\right)^{\gamma_2'}$
  - $\gamma_1' = \{2\gamma_1 \quad \text{for} \quad j = 1; \gamma_1 \quad \text{for} \quad j \neq 1\}$
  - $\gamma_2' = \{2\gamma_2 \quad \text{for} \quad k = 2; \gamma_2 \quad \text{for} \quad k \neq 2\}$

Tests on "physical" models

- $\widehat{D}(a_s) = \frac{1}{\beta_0}\,\mathrm{PV}\int_0^\infty e^{-u/(\beta_0 a_s(s))}\,B(u)\,du$
- $B(u) \;=\; B_1^{UV}(u) + B_2^{IR}(u) + B_3^{IR}(u) + d_0^{PO} + d_1^{PO}u$

$$B_p^{IR}(u) \;=\; \frac{d_p^{IR}}{(p-u)^{1+\tilde\gamma}}\left[1 + \tilde b_1(p-u) + \tilde b_2(p-u)^2 + \ldots\right]$$

$$B_p^{UV}(u) \;=\; \frac{d_p^{UV}}{(p+u)^{1+\bar\gamma}}\left[1 + \bar b_1(p+u) + \bar b_2(p+u)^2\right]$$

The free parameters $(d_1^{UV}, d_2^{IR}, d_2^{IR}, d_0^{PO}, d_1^{PO})$ are fixed by reproducing the known values of $K_n, \;\; n \le 5$

$\Rightarrow \quad d_1^{UV} = -0.015, \quad d_2^{IR} = 3.13$ Beneke & Jamin, 2008

Beneke & Jamin model; Laplace integral for a=0.25

Exact value: $\frac{1}{a}\mathrm{PV}\int_0^\infty e^{-u/a}B(u)du = 0.438518$

N	Stand.Pert.	Sing.soft.u	w_{12}	w_{13}	$w_{1\infty}$	w_{23}
2	0.37286	0.514666	0.415187	0.428041	0.40750	0.436599
3	0.39789	0.366626	0.42055	0.422045	0.418303	0.421829
4	0.41934	0.378312	0.44414	0.440538	0.442288	0.441093
5	0.43309	0.485122	0.444632	0.44456	0.442969	0.444296
6	0.45076	0.43901	0.442506	0.44426	0.442805	0.443279
8	0.48789	0.752828	0.439019	0.440782	0.44122	0.439871
10	0.56348	1.13108	0.439260	0.439472	0.44001	0.43966
12	0.8589	-2.40697	0.438691	0.439678	0.43976	0.43823
14	3.1139	-35.1106	0.438760	0.439202	0.439959	0.43698
15	-5.96303	89.100	0.438674	0.438818	0.440061	0.438250
17	-113.893	352.619	0.438601	0.438372	0.440085	0.425381
18	526.9	-406.753	0.438599	0.438351	0.439981	0.457510

Beneke & Jamin model; Laplace integral for a=0.4

Exact value: $\frac{1}{a} \mathrm{PV} \int_0^\infty e^{-u/a} B(u) du = 0.387625$

N	Stand.Pert.	Sing.soft.u	w_{12}	w_{13}	$w_{1\infty}$	w_{23}
2	0.407522	0.843259	0.497684	0.500514	0.518631	0.509974
3	0.471611	0.404904	0.501175	0.495415	0.512709	0.495926
4	0.559468	0.41256	0.438639	0.473403	0.421621	0.467317
5	0.649575	0.178016	0.423404	0.440652	0.410426	0.423464
6	0.834911	0.921353	0.400717	0.41373	0.40286	0.403333
8	1.71815	0.304534	0.395083	0.385329	0.386413	0.385393
10	6.7374	-14.7768	0.387473	0.386253	0.38156	0.375753
12	61.8365	-97.6926	0.387251	0.379201	0.381715	0.378451
14	1192.37	-77.7393	0.385062	0.372344	0.37905	0.446065
15	-5348.3	-1177.25	0.385254	0.375683	0.375558	0.336412
17	-230325.	-45847.4	0.386237	0.388202	0.364298	0.908281
18	1.66×10^6	233453	0.386069	0.391481	0.357045	-0.603685

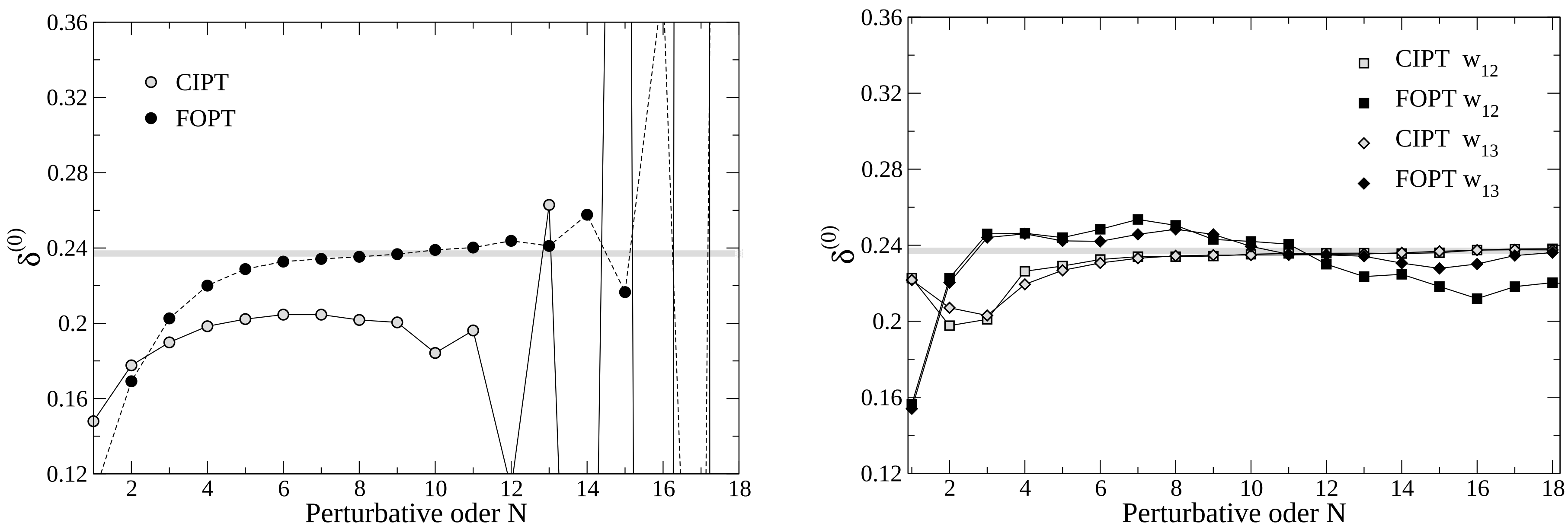
Other models

- More general expression

$$\mathsf{B}(\mathsf{u}) = \sum_{\mathsf{p}\leq -1} \mathsf{B}_{\mathsf{p}}^{\mathrm{UV}}(\mathsf{u}) + \sum_{\mathsf{q}\geq 2} \mathsf{B}_{\mathsf{q}}^{\mathrm{IR}}(\mathsf{u}) + \mathsf{d}_0^{\mathrm{PO}} + \mathsf{d}_1^{\mathrm{PO}}\mathsf{u} + \mathsf{d}_2^{\mathrm{PO}}\mathsf{u}^2$$

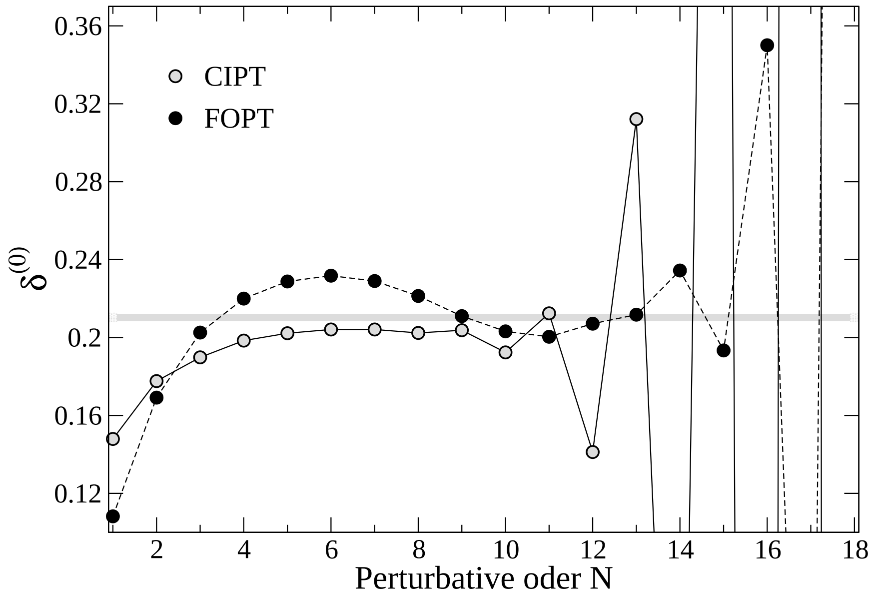
- Extreme cases: impose $\mathsf{d}_2^{\mathrm{IR}} = 1$ or $\mathsf{d}_2^{\mathrm{IR}} = 5$
- Various possibilities were examined numerically
  - How plausible are they physically?

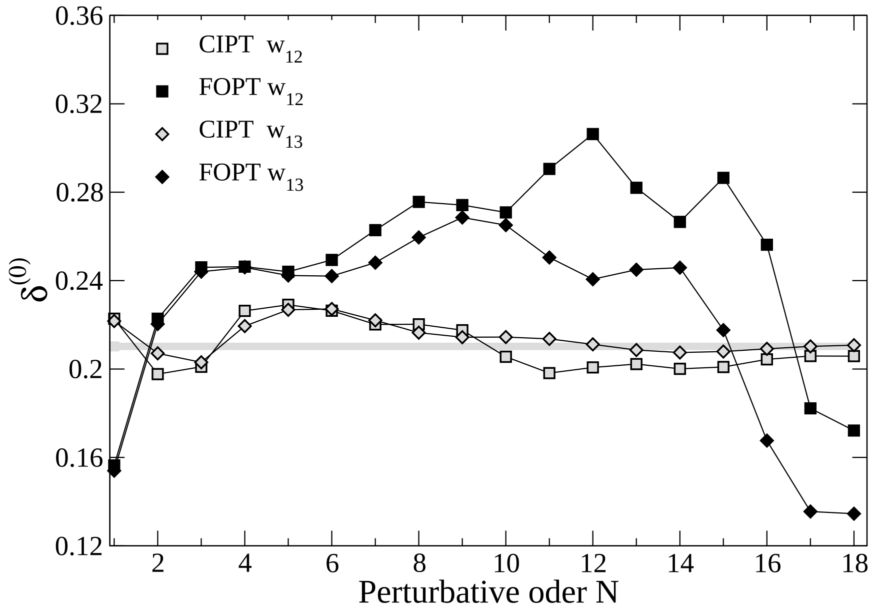
Beneke & Jamin model; $\alpha_s(M_\tau^2) = 0.34$



$\delta^{(0)}$ as a function of the truncation order. Left: standard CIPT and FOPT; right: new CIPT and FOPT

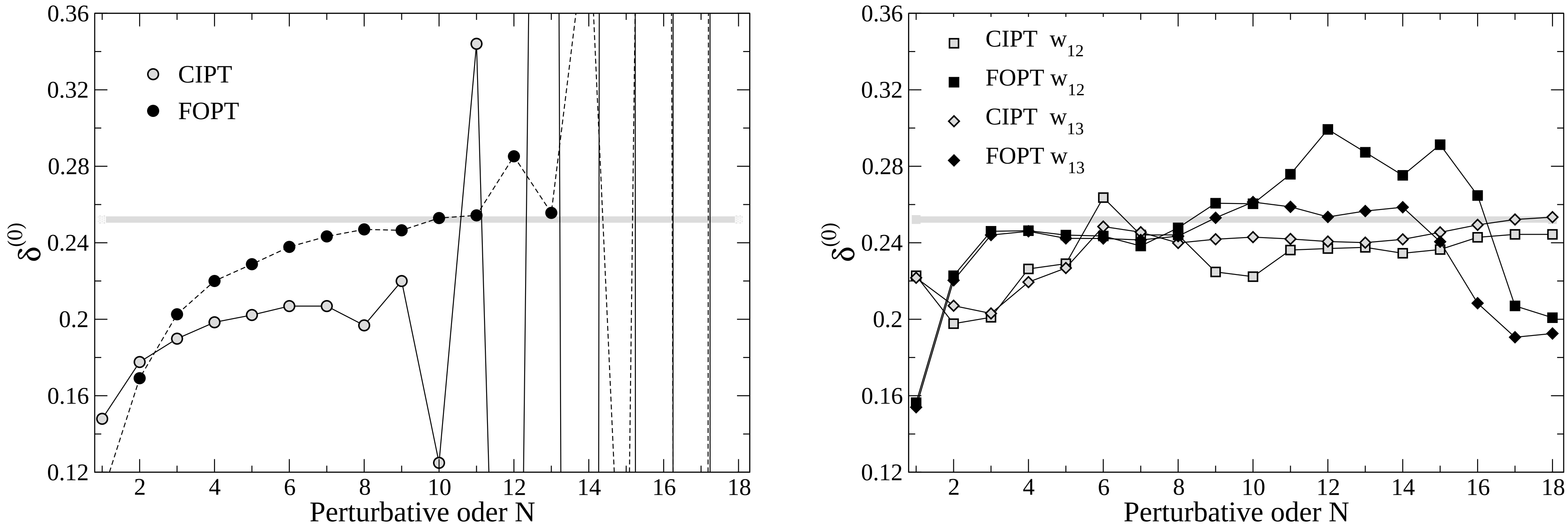
Model with $d_2^{IR} = 1$; $\alpha_s(M_\tau^2) = 0.34$





$\delta^{(0)}$ as a function of the truncation order. Left: standard CIPT and FOPT; right: new CIPT and FOPT

Model with $d_2^{IR} = 5$; $\alpha_s(M_\tau^2) = 0.34$

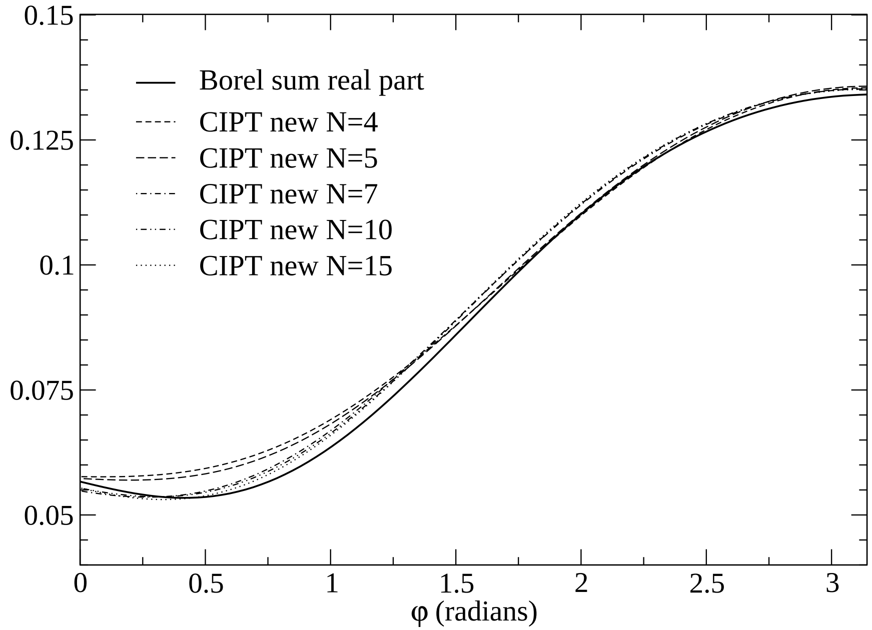


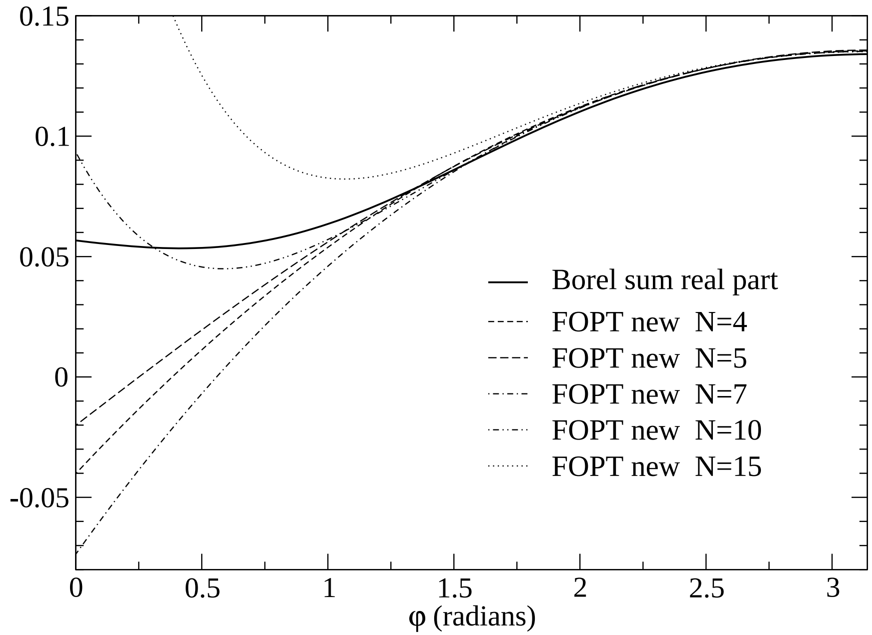
$\delta^{(0)}$ as a function of the truncation order. Left: standard CIPT and FOPT; right: new CIPT and FOPT

Conclusions from the numerical tests

- the expansion based on the optimal conformal mapping is numerically confirmed for all models investigated and all values of a_s
- good results up to rather high orders orders are obtained also with other expansion functions after softening the first singularities
- CIPT gives results better than FOPT

CITP versus FOPT for the optimal expansion





Real part of $\hat{D}(s)$ for $s = m_\tau^2 e^{i\varphi}$ for the optimal expansion based on the conformal mapping w_{12}

Left: new CIPT; right: new FOPT; clearly shows the poor convergence of the expansion of $\alpha_s(s)$ on the circle

Determination of $\alpha_s(M_\tau^2)$

Expansion	$\alpha_s(M_\tau^2)$
Standard expansion in powers of u	0.3425 (0.3421)
Singul. Soften. & expansion in powers of u	0.3176 (0.3178)
Singul. Soften. & expansion in powers of $w_{12}(u)$	0.3198 (0.3199)
Singul. Soften. & expansion in powers of $w_{13}(u)$	0.3212 (0.3211)
Singul. Soften. & expansion in powers of $w_{1\infty}(u)$	0.3186 (0.3187)
Singul. Soften. & expansion in powers of $w_{23}(u)$	0.3197 (0.3197)

$\delta^0 = 0.2042, \quad K_5 = 283$ Beneke, Jamin, JHEP 2008

$\delta^0 = 0.2038, \quad K_5 = 275$ Pich, 2010 (numbers in parentheses)

Average of the new determinations: $\alpha_s(M_\tau^2) = 0.3219 \pm 0.0007$

Adding all the errors (exp. + K_5 + scale): IC & Fischer, EPJC 2009

$$\boxed{\alpha_s(M_\tau^2) = 0.320 \pm 0.011}$$

Conclusions

- An improved systematic perturbative expansion in QCD can be derived using the optimal conformal mapping of the Borel plane

- The new expansion functions are singular at $\alpha_s = 0$ and have divergent expansions in powers of α_s, resembling the expanded function $\widehat{D}$ itself

- Several other choices of "singularity softening" and conformal mappings give consistent results up to rather high orders

- In the CIPT version, they provide a solid theoretical frame for the determination of $\alpha_s(M_\tau^2)$ from τ decays, at the present and future levels of loop calculations