Modified Contour Improved Perturbation Theory

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Introduction

Strong Coupling Values From Different Experiments

	$\alpha_s(M_Z^2)$
R_τ^{V+A} in CIPT	0.1217 ± 0.0017
World average by Bethke (2009)	0.1184 ± 0.0007
Lattice QCD (HPQCD collab., 2008)	0.1183 ± 0.0008
Z_0 decays (Baikov et al, 2008)	0.1190 ± 0.0026
Jet cross section in $p\,\bar{p}$ collisions (2009)	$0.1161^{+0.0041}_{-0.0048}$

Small Tension!

Semihadronic tau decay ratio

$$R_\tau = \frac{\Gamma(\tau \longrightarrow \mathrm{had}\,\nu_\tau\ (\gamma))}{\Gamma(\tau \longrightarrow e^-\ \bar{\mu}_e\,\mu_\tau\ (\gamma))}$$

$$R_\tau = R_\tau^S + R_\tau^V + R_\tau^A$$

$$R_\tau^{V+A} = R_\tau^V + R_\tau^A$$

$$R_\tau = \frac{1 - \mathcal{B}_e - \mathcal{B}_\mu}{\mathcal{B}_e} = \frac{1}{\mathcal{B}_e} - 1.9726 = 3.640 \pm 0.010$$

$$R_\tau^S = 0.1615 \pm 0.0040$$

$$R_\tau^{V+A} = 3.479 \pm 0.011$$

Semihadronic tau decay ratio

$$R_\tau^{V+A} = 3|V_{ud}|^2 S_{ew}(1 + \delta_0 + \delta'_{ew} + \delta_2 + \delta_{NP})$$

$$R_\tau^{V+A} = 3.479 \pm 0.011$$

$$S_{ew} = 1.0198 \pm 0.0006$$

$$\delta'_{ew} = 0.001 \pm 0.001$$

$$\delta_2 = (-4.3 \pm 2.0) \times 10^{-4}$$

$$\delta_{NP} = (-5.9 \pm 1.4) \times 10^{-3}$$ → (of the order of δ_0 s uncertainty)

$$V_{ud} = 0.97418 \pm 0.00027$$

Massless pQCD contribution: $\delta_0 = 0.204 \pm 0.004$

Evaluation of R_τ

$$R_\tau = \int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1 - \frac{s}{M_\tau^2}\right)^2\left(1 + 2\frac{s}{M_\tau^2}\right)\frac{1}{\pi}\mathrm{Im}\,\Pi(s)$$

with $\Pi(s) = |V_{ud}|^2(\Pi_{ud}^V(s) + \Pi_{ud}^A(s)) + |V_{us}|^2(\Pi_{us}^V(s) + \Pi_{us}^A(s))$

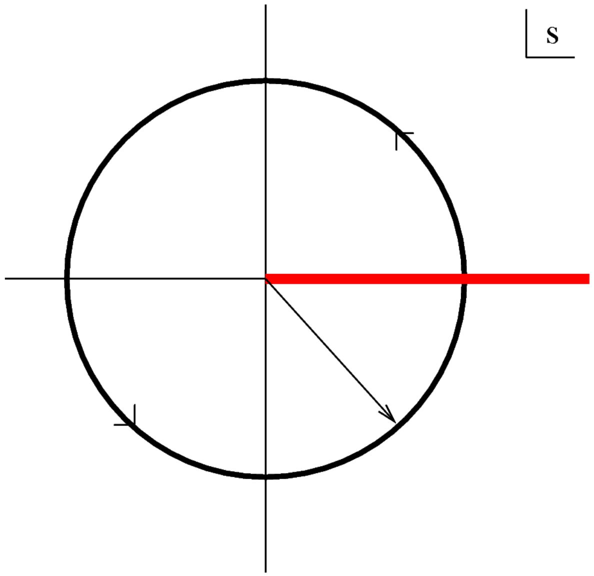
Evaluation of R_τ

$$R_\tau = \int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+2\frac{s}{M_\tau^2}\right)\frac{1}{\pi}\mathrm{Im}\,\Pi(s)$$

with $\Pi(s) = |V_{ud}|^2(\Pi_{ud}^V(s)+\Pi_{ud}^A(s)) + |V_{us}|^2(\Pi_{us}^V(s)+\Pi_{us}^A(s))$

Cauchy Theorem:

$$R_\tau = \frac{-1}{2\pi i}\oint_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+2\frac{s}{M_\tau^2}\right)\Pi(s)$$



Partial Integration:

$$R_\tau = \frac{1}{2\pi i}\oint_{|x|=1} \frac{dx}{x}(1-x)^3(1+x)\,\frac{1}{2}D(-xM_\tau^2)$$

with the Adler function: $D(Q^2) = -Q^2\frac{d\Pi(-Q^2)}{dQ^2}$

Change of notation:

$$R_\tau = \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{dx}{x}(1-x)^3(1+x)\,\frac{1}{2} D(-xM_\tau^2) \longrightarrow$$

$$\longrightarrow \qquad \delta_0 = \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{dx}{x}(1-x)^3(1+x)\,\hat{D}(-xM_\tau^2)$$

using: $R_\tau^{V+A} = 3|V_{ud}|^2 S_{ew}(1+\delta_0+\delta'_{ew}+\delta_2+\delta_{NP})$

and with: $\dfrac{D(Q^2)}{3|V_{ud}|^2 S_{ew}} - 1 \longrightarrow \hat{D}(Q^2)$ Canonically normalized Adler function (massless pQCD)

Contour Improved Perturbation Theory (CIPT) and Fixed Order Perturbation Theory (FOPT)

$$\delta_0 = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x)\, \hat{D}(-xM_\tau^2)$$

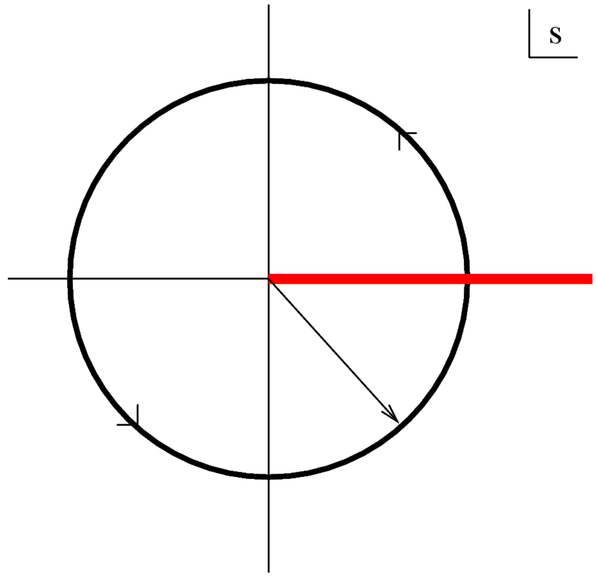
"pinched"

FOPT → $\hat{D}(Q^2) = \sum_{n=1}^{4} a^n(\mu^2) \sum_{m=0}^{n-1} c_{n,m} \log^m(Q^2/\mu^2)$

(m)CIPT

$$\hat{D}^{\mathrm{RG}}(Q^2) = \sum_{n=1}^{4} c_{n,0}\, a^n(Q^2)$$

$$\tilde{D}(Q^2) = \sum_{n=1}^{4} \tilde{c}_n\, \tilde{a}_n(Q^2)$$ (modified)



$a(Q^2) \equiv \alpha(Q^2)/\pi$

Keep in mind:

- The way you use the Renormalization Group is crucial:

  → CIPT and FOPT leads to different results

- Discrimination criterion: behavior under renormalization scale variations

  → CIPT

Modified CIPT

Derivative Expansion for the Adler Function

Instead of the usual power expansion

$$c_1\,a + c_2\,a^2 + c_3\,a^3 + \ldots$$

we use a non-power series of the form:

$$\tilde{c}_1\,\tilde{a}_1 + \tilde{c}_2\,\tilde{a}_2 + \tilde{c}_3\,\tilde{a}_3 + \ldots$$

where the tilde coupling are defined as $\tilde{a}_{m+1} = \frac{(-1)^m}{\beta_0^m\,m!}\frac{d^m a}{d(\log Q^2)^m}$

normalized such that $\tilde{a}_n = a^n + \mathcal{O}(a^{n+1})$

Derivative Expansion for the Adler Function

Instead of the usual power expansion

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we use a non-power series of the form:

$$\tilde{c}_1\, \tilde{a}_1 + \tilde{c}_2\, \tilde{a}_2 + \tilde{c}_3\, \tilde{a}_3 + \ldots$$

$$\begin{aligned}
\tilde{a}_1 &= a \\
\tilde{a}_2 &= -\frac{1}{\beta_0}\beta(a) \\
\tilde{a}_3 &= \frac{1}{2\beta_0^2}\beta'(a)\beta(a) \\
\tilde{a}_4 &= -\frac{1}{6\beta_0^3}\{\beta''(a)\beta(a)^2 + \beta'(a)^2\beta(a)\}
\end{aligned}$$

with $\beta(a) \equiv \dfrac{\partial a}{\partial \log \mu^2} = -(\beta_0\, a^2 + \beta_1\, a^3 + \beta_2\, a^4 + \beta_3\, a^5).$

Derivative Expansion for the Adler Function

- It is just a rearrangement of the series. Both series are in principle equal
- Knowing the first *n* coefficients of the first series we can obtain the first *n* coefficients of the second one
- However, perturbation series in QFT are suppose to be asymptotic at best
- Worse, we are forced to truncate the series at *n* = 4:

$$c_1\, a + c_2\, a^2 + c_3\, a^3 + \ldots + c_n\, a^n \neq \tilde{c}_1\, \tilde{a}_1 + \tilde{c}_2\, \tilde{a}_2 + \tilde{c}_3\, \tilde{a}_3 + \ldots + \tilde{c}_n\, \tilde{a}_n$$

The difference is relevant for big values of *a* !!

If we postulate a skeleton expansion the use of derivatives in the coupling is natural:

$$\mathcal{O}_{\text{skel}}(Q^2) = \int_0^\infty \frac{\mathrm{d}t}{t}\, F_{\mathcal{O}}^{\mathcal{A}}(t)\, \underbrace{a_{\text{pt}}(t e^C Q^2)} + \sum_{n=2}^{\infty} s_{n-1}^{\mathcal{O}} \left[\prod_{j=1}^{n} \int_0^\infty \frac{\mathrm{d}t_j}{t_j} a_{\text{pt}}(t_j e^C Q^2) \right] F_{\mathcal{O}}^{\mathcal{A}}(t_1, \ldots, t_n).$$

$$a(Q^2) = a(Q_0^2) + \log(Q^2/Q_0^2) \underbrace{\left.\frac{da}{d\log Q^2}\right|_{Q^2=Q_0^2}}_{\sim\, \tilde{a}_2} + \frac{1}{2!}\log^2(Q^2/Q_0^2) \underbrace{\left.\frac{d^2 a}{d(\log Q^2)^2}\right|_{Q^2=Q_0^2}}_{\sim\, \tilde{a}_3} + \ldots$$

Numerical Relevance

(extracting α from experiments)

$$\alpha_s^{\mathrm{mCI}}(M_\tau^2) = 0.341 \pm 0.005^{\mathrm{exp}} \pm 0.006^{\mathrm{theo}}$$
$$= 0.341 \pm 0.008$$

$$\alpha_s^{\mathrm{CI}}(M_\tau^2) = 0.347 \pm 0.005^{\mathrm{exp}} \pm 0.014^{\mathrm{theo}}$$
$$= 0.347 \pm 0.015$$

(FOPT: 0.326)

a) significant shift of center value (~ experimental error)

b) lower uncertainty (within the method)

Evaluation

$$\delta_0 = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x)\, \tilde{D}(-x M_\tau^2)$$

with $$\tilde{D}(Q^2) = \sum_{n=1}^{4} \tilde{c}_n \, \tilde{a}_n(Q^2)$$

$$c_{1,0} = 1,\ c_{2,0} = 1.6398,\ c_{3,0} = 6.3710,\ c_{4,0} = 49.076,$$
$$\tilde{c}_1 = 1, \quad \tilde{c}_2 = 1.6398, \quad \tilde{c}_3 = 3.4558, \quad \tilde{c}_4 = 26.385.$$

We evaluate the tilde couplings along a circular contour of radius M_τ^2

Modified CIPT

Extraction of the strong coupling from

$$\delta_0 = 0.204 \pm 0.004$$

δ_0	1	2	3	4	$\sum_{n=1}^{4}$	a
$\mathrm{C}I$	0.1513	0.0308	0.0128	0.0090	0.2038	$0.347/\pi$
$\overline{\mathrm{C}I}$	0.1484	0.0372	0.0104	0.0078	0.2039	$0.341/\pi$

- The extracted value of *a* is reduced by about 2%
- The last term of the series is about 10% smaller

$$c_{1,0} = 1,\ c_{2,0} = 1.6398,\ c_{3,0} = 6.3710,\ c_{4,0} = 49.076,$$
$$\tilde{c}_1 = 1, \quad \tilde{c}_2 = 1.6398, \quad \tilde{c}_3 = 3.4558, \quad \tilde{c}_4 = 26.385.$$

Renormalization Scale and Scheme Dependence

- Renormalization scale dependence:

ξ	$a(\xi M_\tau^2)$	$\delta_0,\ CI$	$\delta_0, \overline{\mathrm{C}I}$
0.7	$0.3831/\pi$	0.2009	0.2020
1	$0.3400/\pi$	0.1984	0.2031
2	$0.2812/\pi$	0.1907	0.1991

Uncertainty in δ_0 :

CIPT → 0.0102
mCIPT → 0.0040

Uncertainty in α_s :
(at the tau scale)

CIPT → 0.013
mCIPT → 0.005

Important Reduction !!!

Taking as a measure of the scale dependence of δ_0 its range of variation when ξ varies between 0.7 and 2

(Renormalization Scheme dependence in CIPT and mCIPT are almost equal)

From the point of view of the standard power series for the Adler function, mCIPT includes higher order terms:

Re-expanding in powers of *a* we obtain non-zero coefficients $c_{n,0}$ for n=5 to 8, e.g. $c_{5,0}$ = 300

How does this coefficient compare to the exact one? ?

But, we can test the method for the known coefficients:

From $c_{1,0}$ and $c_{2,0}$ → $c_{3,0}$= 2.92 (exact: 6.3710)

From $c_{1,0}$, $c_{2,0}$ and $c_{3,0}$ → $c_{4,0}$= 22.7 (exact: 49.076)

Includes significant part !! (a factor ~2.2 smaller in both cases)

(Using the same correction factor we obtain the prediction $c_{5,0}$=300 x 2.2 = 660)

Uncertainty in the extraction of α_S

CIPT: $\Delta\alpha^{\rm scale} = 0.013$ and $\Delta\alpha^{\rm scheme} = 0.004$

$$\alpha_s^{\rm CI}(M_\tau^2) = 0.347 \pm 0.005^{\rm exp} \pm 0.014^{\rm theo}$$
$$= 0.347 \pm 0.015.$$

Modified CIPT: $\Delta\alpha^{\rm scale} = 0.005$ and $\Delta\alpha^{\rm scheme} = 0.004$

$$\alpha_s^{\rm mCI}(M_\tau^2) = 0.341 \pm 0.005^{\rm exp} \pm 0.006^{\rm theo}$$
$$= 0.341 \pm 0.008.$$

• If we consider the difference between CIPT and Modified CIPT as the theoretical uncertainty, we get the same theoretical uncertainty as in mCIPT, i.e. 0.006

Uncertainty in the extraction of α_S

After RG evolution (4-loops) up to the *Z* scale:

$$
\begin{aligned}
\alpha_s^{\mathrm{mCI}}(M_Z^2) &= 0.1211 \pm 0.0006^{\mathrm{exp}} \pm 0.0007^{\mathrm{theo}} \pm 0.0005^{\mathrm{evol}} \\
&= 0.1211 \pm 0.0010,
\end{aligned}
$$

Strong Coupling Values From Different Experiments

	$\alpha_s(M_Z^2)$
R_τ^{V+A} in CIPT	0.1217 ± 0.0017
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Related issue: **Electron-Positron hadronic ratio $R_{e^+e^-}(s)$**

$$\tilde{R}(s) = \frac{1}{2\pi i} \int_{-s-i\varepsilon}^{-s+i\varepsilon} \frac{dz}{z} \tilde{D}(z)$$

- The renormalization group is valid in the Euclidean space
- If we expand R(s) and D(Q^2) in powers of the coupling, they differ in the so-called π^2-terms, which are numerically important

<u>Using the tilde expansion we get a new expansion of R</u>

$$\tilde{R}(s) = \underbrace{\frac{1}{2\pi i} \int_{-s-i\varepsilon}^{-s+i\varepsilon} \frac{dz}{z} a(z)}_{\text{Minkowskian coupling}} + \sum_{n=2}^{4} \frac{(-\tilde{c}_n)}{(n-1)\beta_0 \pi} \mathrm{Im}\left\{\tilde{a}_{n-1}(-s-i\varepsilon)\right\}$$

Old question:
Which is a good expansion parameter for R(s)?

Conclusions

- Modified CIPT: a new method for the calculation of R_τ
- It reduces the renormalization scale dependence by more than 50% → lower uncertainty
- The last term of the series is reduced by about 10%
- We obtain a new expression for the electron-positron hadronic ratio
- The extracted value of α_s from R_τ (V+A) in modified CIPT is lower than in CIPT

$$\alpha_s^{\mathrm{CI}}(M_\tau^2) = 0.347 \pm 0.015 \longrightarrow \alpha_s^{\mathrm{mCI}}(M_\tau^2) = 0.341 \pm 0.008$$ (1.8% lower)

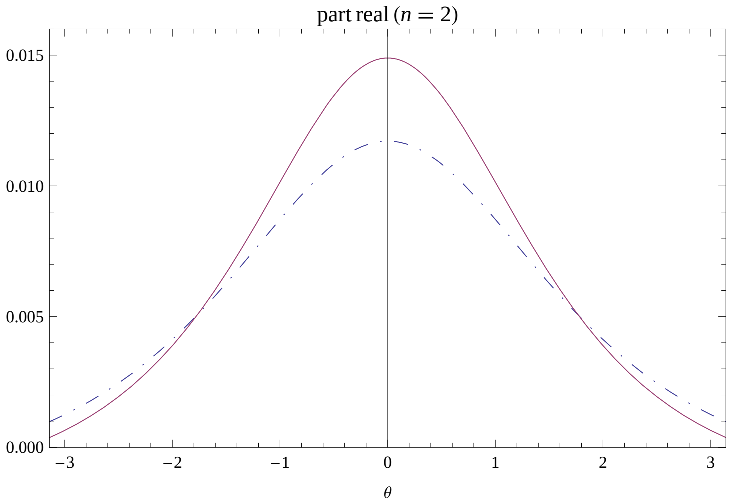
or

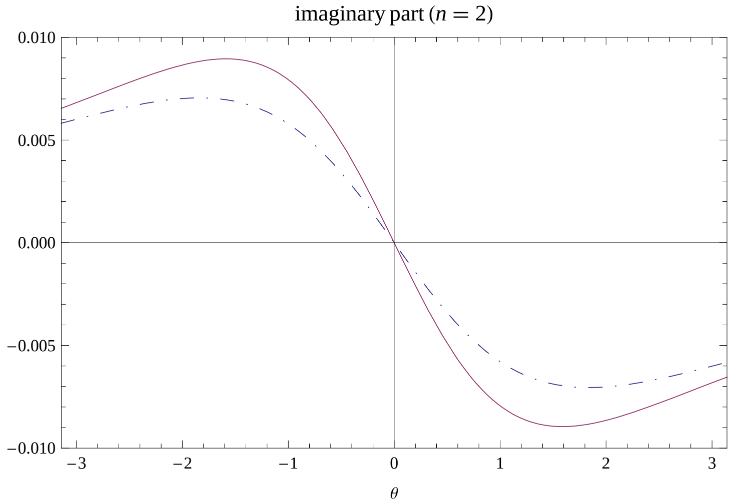
$$\alpha_s^{\mathrm{CI}}(M_Z^2) = 0.1217 \pm 0.0017 \longrightarrow \alpha_s^{\mathrm{mCI}}(M_Z^2) = 0.1211 \pm 0.0010$$ (0.5% lower)

Work to be done: apply this approach in other sum rules!

Backup slides

Comparison of $\tilde{a}_n$ and a^n

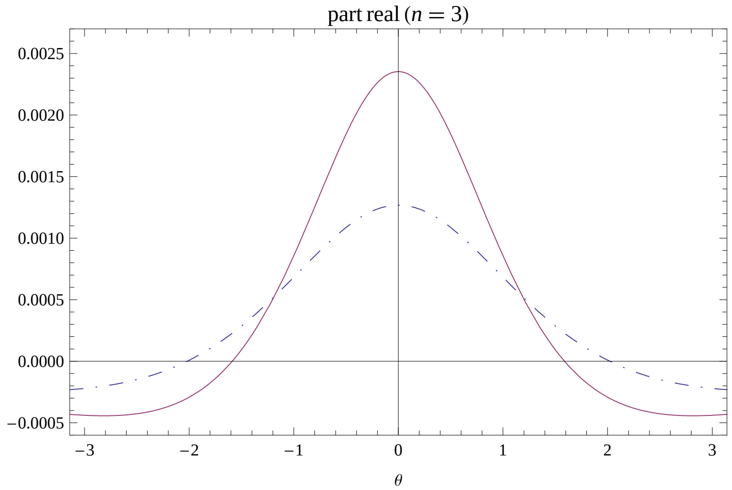


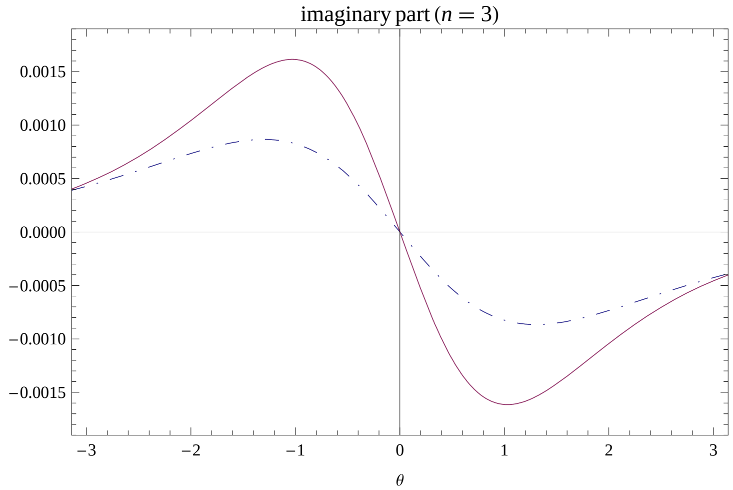


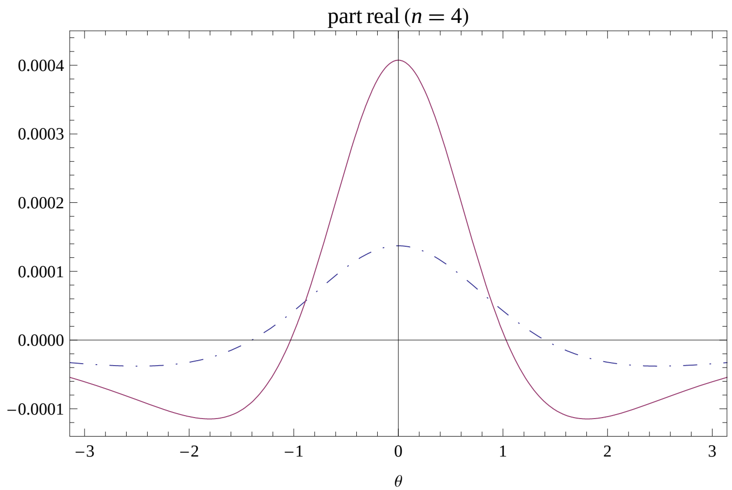
Real and Imaginary part of $\tilde{a}_n(M_\tau^2 e^{i\theta})$ (solid line)
as a function of θ compared to $a^n(M_\tau^2 e^{i\theta})$ (dashed line)

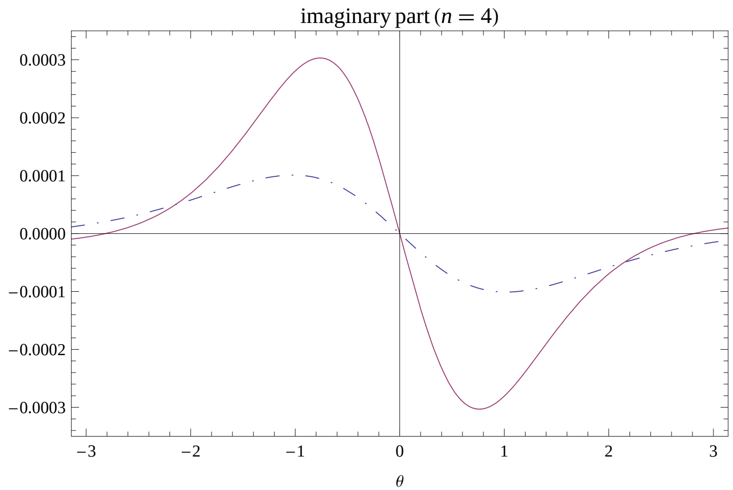
In both cases we take $a(M_\tau^2) = 0.340/\pi$

Comparison of $\tilde{a}_n$ and a^n









Modified CIPT

Extraction of the strong coupling from

$$\delta_0 = 0.204 \pm 0.004$$

Guess of the next perturbative coefficient using Fast Apparent Convergence:

$$c_{5,0} = 275$$
$$\tilde{c}_5 = -25.4$$

δ_0	1	2	3	4	5	$\sum_{n=1}^{4}$	$\sum_{n=1}^{5}$	a
CI	0.1513	0.0308	0.0128	0.0090	(0.0038)	0.2038	(0.2077)	$0.347/\pi$
$\overline{\mathrm{C}I}$	0.1484	0.0372	0.0104	0.0078	(−0.0001)	0.2039	(0.2037)	$0.341/\pi$

↓

!!!