M-brane dynamics and free energy of $D = 3$ superconformal field theories

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SUSY2014 Manchester
21 July 2014

Based on arXiv:1401.3595
with Dongmin Gang (KIAS) and Sangmin Lee (SNU)

AdS/CFT

- AdS/CFT correspondence is a remarkable proposal which relates gravity and quantum field theory.
- It proposes that classical gravity action of AdS space with appropriately chosen boundary condition is equal to some expectation value of strongly coupled quantum field theory.

DOF counting

- In String theory and M-theory, branes generically exhibit gauge field dynamics.
- In string theory, D-branes give rise to Yang-Mills theory, apparently with N^2 scaling of DOF.
- For M-theory branes, microscopic understanding is less clear.
- It is well known that N M2-branes carry $N^{3/2}$ dofs, while for M5-branes dofs scale like N^3.

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Scale symmetry of M-theory

- 11d sugra

$$S = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}R - \frac{1}{2}G \wedge *G - \frac{1}{6}C \wedge G \wedge G$$

- Homogeneous under $g \to \lambda^2 g$, and $C \to \lambda^3 C$. $S \to \lambda^9 S$ but e.o.m. is the same.
- Consequently,

$$ds^2 = r^2 ds_0^2, \quad G = r^3 G_0, \quad S = r^9 S_0$$

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D.o.f. on M2 and M5-branes

- Recall flux quantization,

$$N_{M2} = \int_{X_7} *G \sim r^6$$
$$N_{M5} = \int_{X_4} G \sim r^3$$

- If we substitute them into $S = r^9 S_0$, we get

$$S = N^{3/2} S_0 \quad \text{(M2-branes)}$$
$$S = N^3 S_0 \quad \text{(M5-branes)}$$

  which can be applied to any gravity computation, e.g. gravitational free energy. In particular also for M-brane theories with dimensional reduction.

M2-branes theory

- Near horizon limit of M2-branes in flat space or at tip of CY4 singularity

$$ds^2 = r_0^2 \left[ds^2(AdS_4) + ds^2(SE_7) \right], \quad G \sim r_0^3 \mathrm{vol}(AdS_4)$$

- Standard prescription gives for large N

$$F = N^{3/2} \sqrt{\frac{2\pi^6}{27 \mathrm{vol}(X^7)}}$$

- Can we confirm this using genuine QFT computation?
  - If the answer is yes, can we intuitively understand $N^{3/2}$ behavior?

ABJM model (2008)

- 2+1d superconformal field theory for M2-brane proposed by Aharony, Bergman, Jafferis and Maldacena arXiv:0806.1218
- Localization technique can be applied: partition function, Wilson loops can be computed exactly when the theory is put on S^3. [Kapustin, Willett, Yaakov (2009)]

ABJM matrix model

- Z as a function of N_1, N_2, k and an ordinary integral over eigenvalues.

$$Z_{ABJM} = \frac{1}{N_1! N_2!} \int \prod_i^{N_1} \frac{d\mu_i}{2\pi} \prod_j^{N_2} \frac{d\nu_j}{2\pi} e^{\frac{ki}{4\pi}(\sum \mu_i^2 - \sum \nu_j^2)}$$
$$\times \quad \frac{\prod_{i<j}(2\sinh(\mu_i - \mu_j))^2 \prod_{i<j}(2\sinh(\nu_i - \nu_j))^2}{\prod_{i,j}(2\cosh(\mu_i - \nu_j)/2)^2}$$

- Drukker, Marino, Putrov (2010,2011): The integral at hand is related to Lens space matrix model whose exact solution is already known.
- Shown that $F \equiv -\log|Z| \sim k^{1/2} N^{3/2}$.

CS matrix model and eigenvalue dynamics

- In the large N limit one can employ saddle point approximation, and the eigenvalue dynamics leads to $N^{3/2}$ behavior or free energy with correct coefficient for ABJM model (Herzog, Klebanov, Pufu, Tesileanu arXiv:1011.5487).
- Also applied to $\mathcal{N} = 2$ systems in Martelli, Sparks arXiv:1102.5289; S. Cheon, H. Kim, NK arXiv:1102.5565; Jafferis, Klebanov, Pufu, Safdi arXiv:1103.1181

Eigenvalues as fermion gas

- Developed by Marino and Putrov arXiv:1110.4066
- Through astute manipulation, one can rewrite

$$Z_{ABJM} = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \int \frac{d^N x}{(2\pi k)^N} \frac{1}{\prod_i 2\cosh\left(\frac{x_i}{2}\right) 2\cosh \frac{x_i - x_{\sigma(i)}}{2k}}$$

- Total antisymmetrization is manifest, and the system consists of just N identical fermions with single-particle density matrix

$$\rho = \frac{1}{2\pi k} \frac{1}{\left(2\cosh \frac{x_1}{2}\right)^{1/2}} \frac{1}{\left(2\cosh \frac{x_1 - x_2}{2k}\right)^{1/2}} \frac{1}{\left(2\cosh \frac{x_2}{2}\right)^{1/2}}$$

which comes from

$$\rho = e^{-U(q)/2} e^{-T(p)} e^{-U(q)/2}, \quad T(x) = U(x) = \log\left(2\cosh x/2\right)$$

Free energy of matrix model

- For large N, $E = (|p| + |q|)/2$ and $n \sim E_{fermi}^2$.
- From general property of grand-canonical ensemble $F \sim N^{\frac{s+1}{s}}$ if $n(E) \sim E^s$. For us $s = 2$ (1-dim)
  - $F = J(\mu) - \mu N(\mu)$ and $N = \frac{\partial J}{\partial \mu}$
  - So if $n \sim E^s$, $J \sim \mu^{s+1}$ and $N \sim \mu^s$
- Or more simply, F scales as total energy of ground state for N particles: $F \sim \int_0^N E dn \sim N^{3/2}$.

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M5-branes

- 6d theory with $(2,0)$ supersymmetry, and still no Lagrangian description for multiple M5-branes.
- A number of proposals to account for N^3, in particular through $4+1d$ super Yang-Mills after reduction on S^1.

M5 on supersymmetric-cycle

- M5-branes wrapped on supersymmetric cycles lead to nontrivial SCFT in lower dimensions
- 2-cycle as Riemann surface with punctures: $D = 4$ $N = 2$ SCFT of class S.
- Manipulation on Riemann surface and the punctures give S-duality operation on gauge theory side. D. Gaiotto, arXiv:0904.2715
- AGT relation (Alday, Gaiotto, Tachikawa arXiv:0906.3219):
    - Nekrasov partition function of $N = 2$ SCFT is equal to correlation functions of Liouville theory on Riemann surface.

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3d/3d relation and M5 on 3-cycle

- A precise relation between SCFT in 3d, and another purely bosonic QFT also in 3d.
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- Can be derived using localization prescription, by putting 5d SYM on $M = S^2 \times M_3$. (S. Lee and Yamazaki, arXiv:1305.2429; Cordova and Jafferis, arXiv:1305.2891)

Dictionary for 3d/3d relation

(Taken from Dimofte, Gabella, Goncharov arXiv:1301.0192)

$T_{\mathfrak{g}}[M, \Pi]$		M, $G_{\mathbb{C}}$ connections
coupling to $T_{\mathfrak{g}}[\partial M]$	$\longleftrightarrow$	polarization Π for $\mathcal{P}_{G_{\mathbb{C}}}(\partial M)$
rank of flavor symmetry group	$\longleftrightarrow$	$\frac{1}{2}\dim_{\mathbb{C}} \mathcal{P}_{G_{\mathbb{C}}}(\partial M)$
SUSY parameter space on $\mathbb{R}^2 \times S^1$	$\longleftrightarrow$	flat connections extending to M, $\mathcal{L}_{G_{\mathbb{C}}}(M)$
flavor Wilson and 't Hooft ops	$\longleftrightarrow$	quantized algebra of functions on $\mathcal{P}_{G_{\mathbb{C}}}(\partial M)$
Ward id's for line operators	$\longleftrightarrow$	quantization of $\mathcal{L}_{G_{\mathbb{C}}}(M)$
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Localization on squashed 3-sphere

- One can put $\mathcal{N} = 2$, $D = 3$ theories on squashed S^3, defined as

$$b^2(x_1^2 + x_2^2) + b^{-2}(x_3^2 + x_4^2) = 1$$

- The partition function involves double sine function

$$s_b(x) = \prod_{m,n \in \mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + Q/2 - ix}{mb + nb^{-1} + Q/2 + ix}, \quad Q = b + 1/b$$

Gravity dual of CFT on squashed S^3

- Martelli, Passias, Sparks arXiv:1110.6400, 1111.6930 found solutions in Euclidean $D = 4$ $\mathcal{N} = 2$ gauged supergravity where the metric becomes that of squashed 3-sphere on the boundary.
- Computed gravitational free energy, and found

$$F_b = \frac{(b + 1/b)^2}{4} F_{b=1}$$

and verified it in all Chern-Simons-matter models which are field theory duals of $AdS_4 \times SE_7$.

M5 on Σ_3 in CY3

- Constructed in Gauntlett, NK, Waldram (2000)

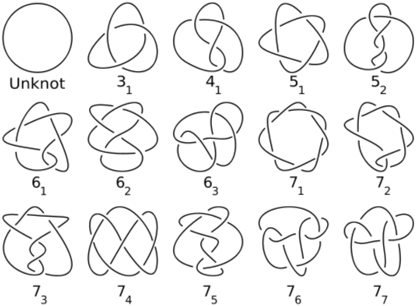
$$ds_{11}^2 = \frac{2^{2/3}(1+\sin^2\theta)^{1/3}}{g^2}\Bigg[ds^2(AdS_4) + ds^2(M) + \frac{1}{2}\left(d\theta^2 + \frac{\sin^2\theta}{1+\sin^2\theta}d\phi^2\right) + \frac{\cos^2\theta}{1+\sin^2\theta}d\tilde{\Omega}_2\Bigg]$$

- One may relate parameter g with N_{M5} through flux quantization (on squashed S^4) and consider dimensional reduction to 4d to get

$$F^{\text{gravity}} = \frac{N^3}{12\pi}\left(b+\frac{1}{b}\right)^2 \text{vol}(M)\,,$$

H^3 as knot complements

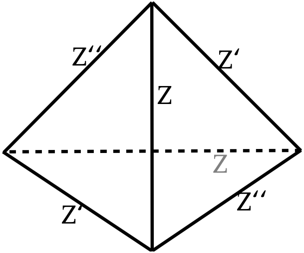
- We take $=S^3\backslash K$, and simplest knots are as follows (Wikipedia)



- Knots that lead to hyperbolic space are $4_1, 5_2, 6_1, 6_2, 6_3, 7_2, 7_3$ etc.
- The volumes for hyperbolic metric are topological invariants.
- For instance $\mathrm{vol}(S^3\backslash 4_1) = 2\mathrm{Im}\left(\mathrm{Li}_2(e^{\frac{i\pi}{3}})\right) = 2.02988\cdots$

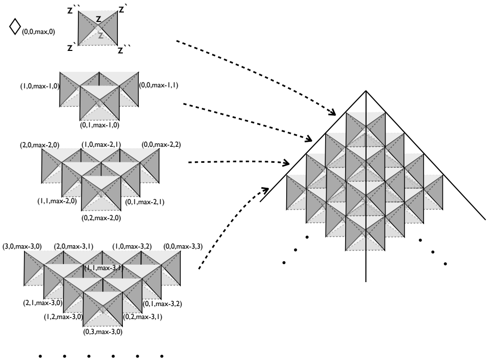
Flat connections on boundary of tetrahedron

- T. Dimofte, D. Garoufalidis etc. 2011-2013
- To quantize, we consider phase space of flat connections on boundary of M.
- Tetrahedron: $Z + Z' + Z'' = \pi i$, parametrizing $\mathbb{C}^2$.
- $\{Z, Z'\} = \{Z', Z''\} = \{Z'', Z\} = 1$
- Partition function for $N = 2$ is quantum dilogarithm, which is wavefunction for $e^{Z''} + e^{-Z} = 1$.



Octahedral decomposition

- A hyperbolic space is first triangulated into k tetrahedra. Then each of them is further decomposed into $1+(1+2)+(1+2+3)+\cdots+(1+2+\cdots+N-1) = N(N^2-1)/6$ octahedra.



$PGL(N)$ CS partition function

- In terms of the gluing data (A_N, B_N) for knot complement ($b \to 1/b$ symmetry can be shown). $M_N = kN(N^2-1)/6$

$$Z_N^{\mathrm{CS}}[M] = \frac{1}{\sqrt{\det B_N}} \int \frac{d^{\mathcal{M}_N}X}{(2\pi\hbar)^{\mathcal{M}_N/2}} \prod \psi_\hbar(X) \times$$
$$\exp\left[-\frac{1}{\hbar}(i\pi + \frac{\hbar}{2})X^T B_N^{-1}\nu_N + \frac{1}{2\hbar}X^T B_N^{-1} A_N X\right]$$

- Can perform perturbative expansion for small $\hbar = 2\pi i b^2$.
- Holographic formula implies the asymptotic series becomes convergent at large N.

$$F = -\log|Z| \to (b+1/b)^2 \mathrm{vol}(S^3 \backslash K)\frac{N^3}{12\pi}$$

Computation

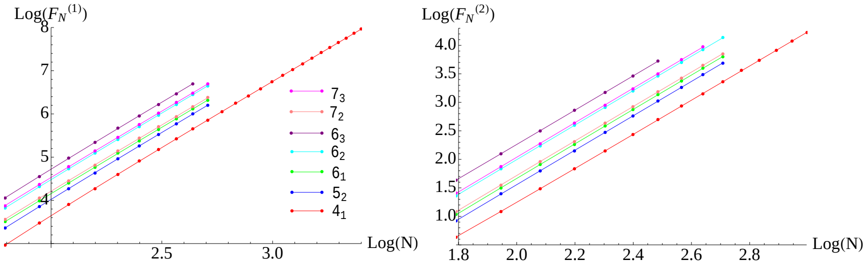
- Saddle point expansion and expansion in terms of Feynman diagrams.

$$F = \frac{i}{\hbar} F^{(0)} + F^{(1)} + \frac{\hbar}{i} F^{(2)} + \cdots$$

- Tree level $F^{(0)}$ is exactly $N(N^2 - 1)/(12\pi)\mathrm{vol}(M)$
- Otherwise cannot take large N limit analytically: resort to numerical computation.
- 2-loop has 7 diagrams, and at 3-loop 40 terms.
- Used program SnapPy and checked up to $N \sim 20$.

results

- Log-log plot of 1 and 2-loop coefficients



- 3-loop coefficients are decreasing as N increases.

results

K	$\mathrm{vol}(S^3 \backslash K)$	$\pi F_N^{(1)\prime\prime\prime}$	(N)	$4\pi^2 F_N^{(2)\prime\prime\prime}$	(N)
$\mathbf{4}_1$	2.02988	2.03001	(27)	2.02898	(17)
$\mathbf{5}_2$	2.82812	2.82828	(12)	2.82674	(12)
$\mathbf{6}_1$	3.16396	3.20648	(12)	3.15574	(12)
$\mathbf{6}_2$	4.40083	4.40364	(12)	4.39929	(12)
$\mathbf{6}_3$	5.69302	5.69464	(11)	5.68799	(9)
$\mathbf{7}_2$	3.33174	3.56613	(12)	3.27455	(12)
$\mathbf{7}_3$	4.59213	4.58680	(12)	4.58331	(11)

Discussion

- M5-brane holography and 3d/3d relation verified. N^3 scaling as well as correct coefficients of perturbative expansion w.r.t. squashing parameter.
- N^3 scaling from discretization (triangulation) of $\Sigma_3 = H^3$ into (approx) N^3 tetrahedra.
- Taking large N limit analytically?
- Defects from probe M2 or M5-branes (e.g. Wilson loop)