Renormalization group flows and the Weyl consistency conditions

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CP3 Origins

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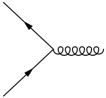
Antipin, Gillioz, Krog, EM & Sannino (2013) arXiv:1306.3234
EM & Schrock (2014) arXiv:1403.3058

SUSY 14
University of Manchester

Running coupling constants

- g_s, the coupling constant in QCD, is the interaction strength between a gluon and a quark.



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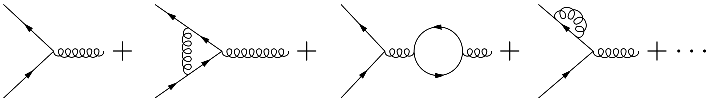

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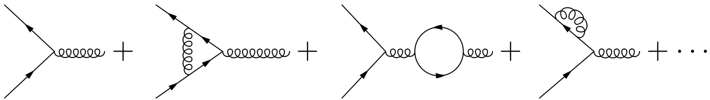

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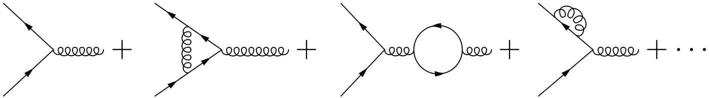

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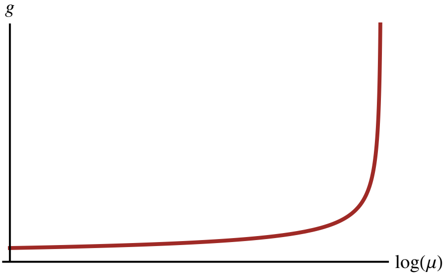
- Each diagram is evaluated at a renormalization energy scale μ.
- The dependence on μ is given by the beta function $\beta_g = \mu\frac{\mathrm{d}g}{\mathrm{d}\mu}$

Running to or from



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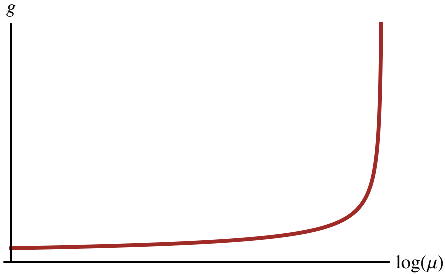

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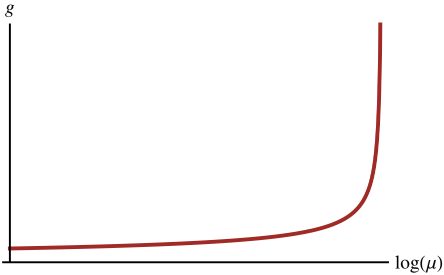

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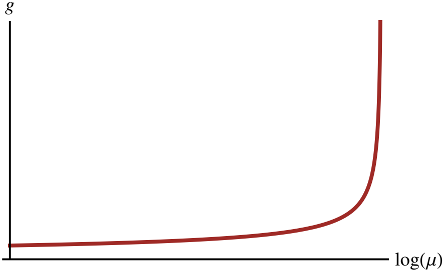

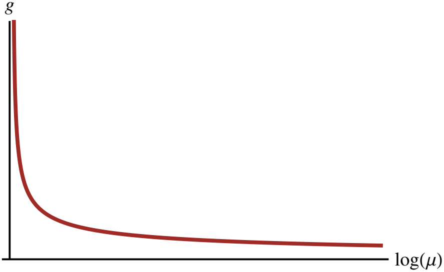

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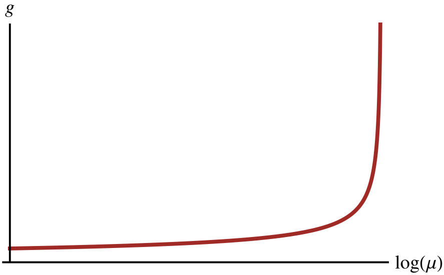

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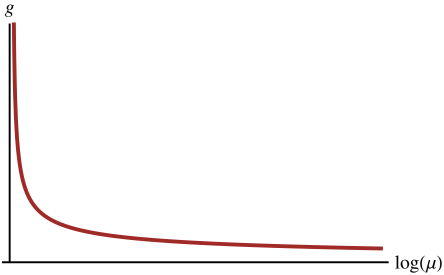

- Asymptotically free
- $g \to 0$ at high energy

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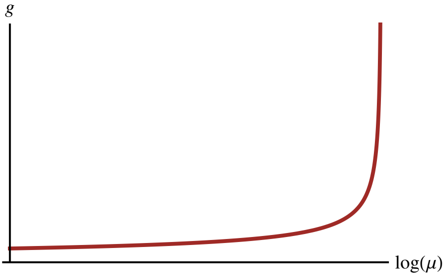

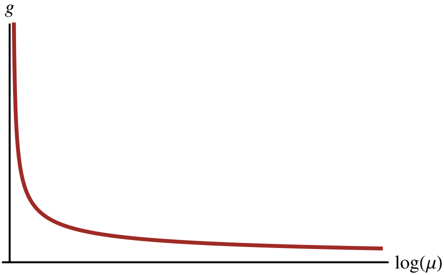

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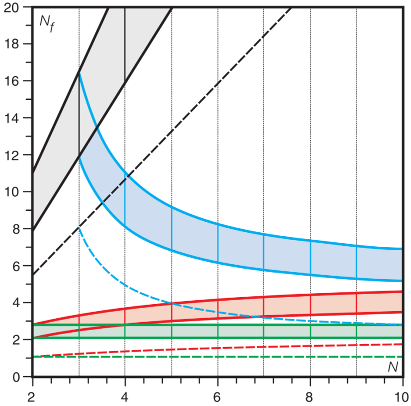
Phase diagram of quantum field theories



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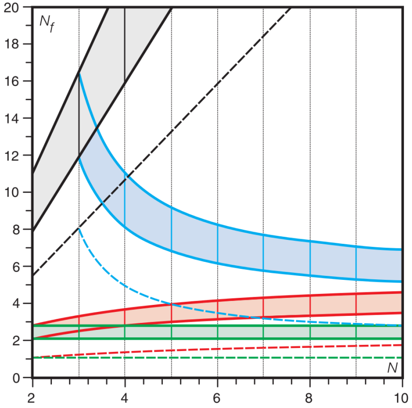

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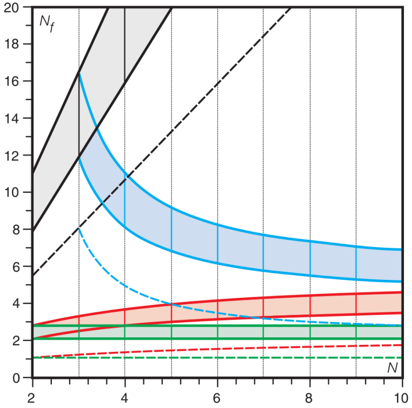

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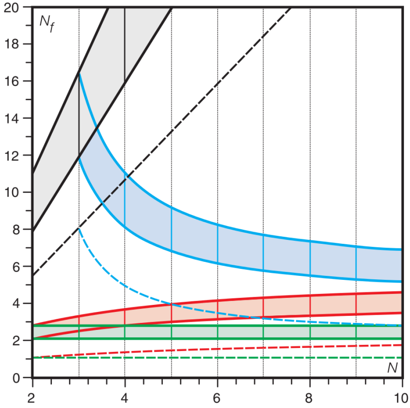

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It all depends on the features of the beta functions!
(And non-perturbative effects.)

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and have

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- and we assume $\mu_\phi \ll \mu$

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- And compute the beta functions to 2 loops in perturbation theory.

Beta functions

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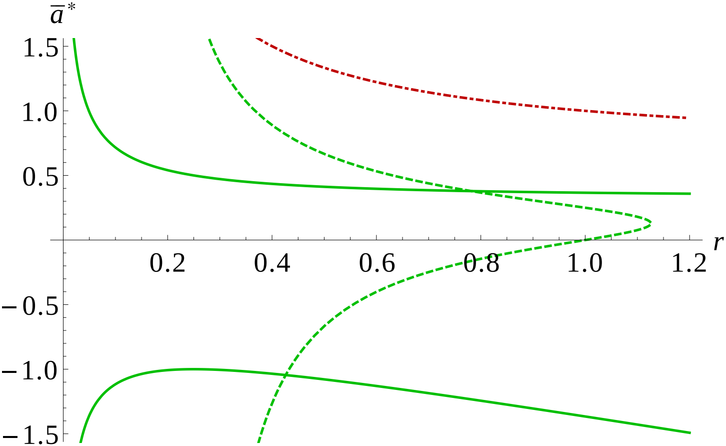
- Non-trivial solutions only for $n = 2$.



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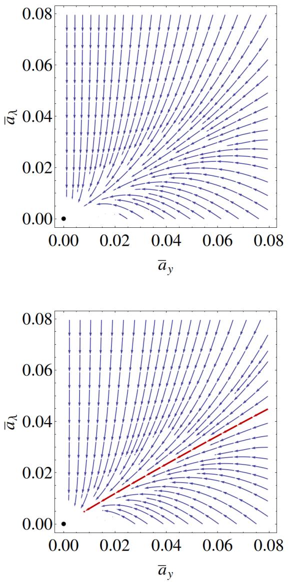

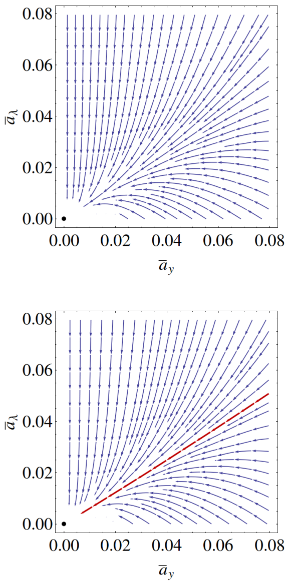

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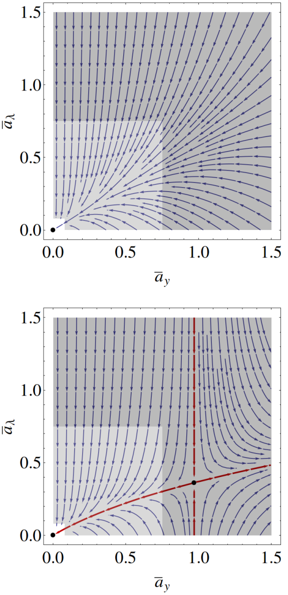

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- Each order should only make "small" corrections.
- Non-perturbative phenomena – condensation and bound states.
- Unclear how to choose n and k.

The Weyl consistency conditions

$$\frac{\partial^2 \tilde{a}}{\partial g_i \partial g_j} \approx \frac{\partial \chi^{jk} \beta_k}{\partial g_i} \approx \frac{\partial \chi^{ik} \beta_k}{\partial g_j} \tag{12}$$

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Thus, to preserve Weyl symmetry in a gauge-Yukawa theory, we must use

- the gauge beta function to $n+2$ loops,
- the Yukawa beta function to $n+1$ loops,
- the quartic beta function to n loops.

Generic beta functions

Generically, beta functions in a gauge-Yukawa theory have the form

$$\beta_{a_g} = a_g^2\big(b_1(a_g) + b_2(a_g, a_y) + b_3(a_g, a_y, a_\lambda)\big) \tag{13}$$

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Which is automatically in line with the Weyl consistency conditions!

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- A new principle is needed for perturbation theory to be trustable.
- The Weyl consistency conditions are required by conformal symmetry and provide such a principle.
- To satisfy them, we must adopt the 321 counting scheme at the lowest order in the beta functions.

Perturbative toy model

Antipin, Mojaza & Sannino (2011) arXiv:1107.2932

$$\mathcal{L} = \mathcal{L}_K(G_\mu, \lambda_m, Q, \tilde{Q}, H) + \left(y_H Q H \tilde{Q} + \text{h.c}\right) - u_1 \left(\text{Tr}\left[HH^\dagger\right]\right)^2 - u_2 \text{Tr}\left[(HH^\dagger)^2\right], \quad (16)$$

Fields	$[SU(N_{TC})]$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_V$	$U(1)_{AF}$
λ_m	Adj	1	1	0	1
Q	$\square$	$\overline{\square}$	1	$\frac{N_f - N_{TC}}{N_{TC}}$	$-\frac{N_{TC}}{N_f}$
$\tilde{Q}$	$\overline{\square}$	1	$\square$	$-\frac{N_f - N_{TC}}{N_{TC}}$	$-\frac{N_{TC}}{N_f}$
H	1	$\square$	$\overline{\square}$	0	$\frac{2N_{TC}}{N_f}$
G_μ	Adj	1	1	0	0

Table: The field content of the toy model and the related symmetries

Modified fixed points

Antipin, Di Chiara, Mojaza, EM & Sannino (2012) arXiv:1205.6157

Antipin, Gillioz, EM & Sannino (2013) arXiv:1303.1525

We investigate this model in the Veneziano limit of large N_{TC} and large N_f, with $x = \frac{N_f}{N_{TC}}$ fixed and rescaled couplings

$$a_g = \frac{g^2 N_{TC}}{(4\pi)^2}, \quad a_H = \frac{y_H^2 N_{TC}}{(4\pi)^2}, \quad z_1 = \frac{u_1 N_f^2}{(4\pi)^2}, \quad z_2 = \frac{u_2 N_f}{(4\pi)^2}. \tag{17}$$

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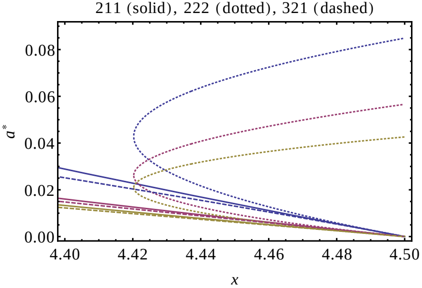
To illuminate the importance of the beta function counting scheme, we here reproduce the analysis done in three different ones

- **211** to 2 loops in a_g, 1 loop in a_H and 1 loop in z_2.
- **222** to 2 loops in a_g, 2 loops in a_H and 2 loops in z_2.
- **321** to 3 loops in a_g, 2 loops in a_H and 1 loop in z_2.

Modified fixed points

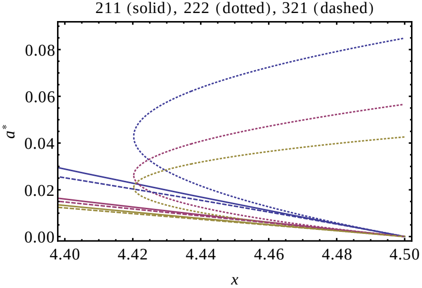
We trust perturbation theory if the changes order by order are "small"

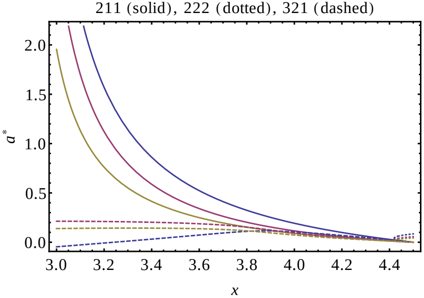


Fixed point values for a_g (blue), a_H (red) and z_2 (yellow)

Modified fixed points

We trust perturbation theory if the changes order by order are “small”

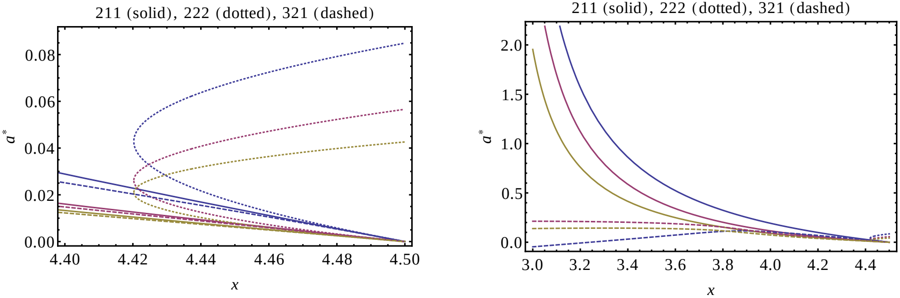




Fixed point values for a_g (blue), a_H (red) and z_2 (yellow)

Modified fixed points

We trust perturbation theory if the changes order by order are “small”



Fixed point values for a_g (blue), a_H (red) and z_2 (yellow)

The 321 scheme does not introduce new spurious fixed points, and keeps the values from diverging.