# Gauge invariance in PT-symmetric scalar QED

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1. Non-Hermitian models and PT symmetry
2. Free coupled scalars
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J. A, P. Millington and D. Seynaeve - PRD 2017
J. A, J. Ellis, P. Millington and D. Seynaeve - PRD 2018, PRD 2019

# Non-Hermitian models and PT symmetry

No need for Hermiticity, in order to have real energies
$\mathcal{PT}$ symmetry enough *Bender - Contemp. Phys. 2005*
$\rightarrow$ Non-Hermitian extension of the SM?
$\rightarrow$ Alternative description of the Higgs sector?

*Examples of $\mathcal{PT}$-symmetric field theories:*

negative quartic interaction $-g\phi^4$ *Shalaby, Al-Thoyaib - PRD 2010*
imaginary cubic interaction $ig\phi^3$ *Bender, Branchina, Messina - PRD 2012 and 2013*
Liouville-style model $\exp(ig\phi)$ *Bender, Hook, Mavromatos, Sarkar - PRL 2014*
QED with mass term $\overline{\psi}\gamma^5\psi$ *Alexandre, Bender, Millington - JHEP 2015*
imaginary Yukawa interaction $h\overline{\psi}\gamma^5\psi$ *Korchin, Kovalchuk - PDR 2016*
lattice fermions *Chernodub - J. Phys. A 2017*

# Free coupled scalars

*Alexandre, Millington, Seynaeve - PRD 2017*

$$\mathcal{L} = \partial_\nu \phi_1^\star \partial^\nu \phi_1^\star + \partial_\nu \phi_2^\star \partial^\nu \phi_2 - \begin{pmatrix} \phi_1^\star & \phi_2^\star \end{pmatrix} \begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

- non-Hermitian mass term $\rightarrow$ complex Lagrangian density
- Mass matrix eigenvalues

$$M_\pm^2 = \frac{m_1^2 + m_2^2}{2} \pm \frac{1}{2}\sqrt{|m_1^2 - m_2^2|^2 - 4\mu^4}$$

$\rightarrow$ real for $2\mu^2 \leq |m_1^2 - m_2^2|$

- Eigen vectors not perpendicular $e_1^\star \cdot e_2 \neq 0$

Real energies for appropriate set of parameters

# Discrete symmetries

- Parity and time reversal for $\Phi(t,\mathbf{x}) \equiv \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$

$$\begin{aligned} \mathcal{P}: \quad & \Phi(t,\mathbf{x}) \longrightarrow \Phi'(t,-\mathbf{x}) = P\,\Phi(t,\mathbf{x}) \\ \mathcal{T}: \quad & \Phi(t,\mathbf{x}) \longrightarrow \Phi'(-t,\mathbf{x}) = T\,\Phi^\star(t,\mathbf{x}) \end{aligned}$$

- $\mathcal{PT}$-symmetry realised for

$$T = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad P = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$\rightarrow$ $\phi_2$ is a pseudo scalar

- Eigen vectors orthogonal with $\mathcal{PT}$-inner product

$$e_1^{\mathcal{PT}} \cdot e_2 = 0$$

# Equations of motion

$$\frac{\delta S}{\delta \Phi^\dagger} = 0 \quad \text{and} \quad \left(\frac{\delta S}{\delta \Phi^\dagger}\right)^\dagger = 0$$

OR

$$\frac{\delta S}{\delta \Phi} = 0 \quad \text{and} \quad \left(\frac{\delta S}{\delta \Phi}\right)^\dagger = 0$$

Not identical $\mu^2 \to -\mu^2$
But related by $\mathcal{PT}$ symmetry

Observables depend on $\mu^4$
$\to$ physics independent of the choice

Two sets of equations of motion related by $\mathcal{PT}$ symmetry and physically equivalent

# Conservation laws

Individual currents

$$j_1^\nu = i\left(\phi_1^\star \partial^\nu \phi_1 - \phi_1 \partial^\nu \phi_1^\star\right) \quad \text{and} \quad j_2^\nu = i\left(\phi_2^\star \partial^\nu \phi_2 - \phi_2 \partial^\nu \phi_2^\star\right)$$

- Lagrangian invariant under phase transformation $\Phi' = \exp(i\alpha\Phi)$

$\rightarrow$ Noether current $= j_1^\nu + j_2^\nu$

- BUT conserved current $= j_1^\nu - j_2^\nu \rightarrow$ phase transformation

$$\Phi' = \exp\left(i\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi\right)$$

$\rightarrow$ source/sink behaviour

The conserved current is not the Noether current

# Gauged model

Two options *Alexandre, Ellis, Millington, Seynaeve - PRD 2019*

1. Gauge field coupled to conserved current $j_1^\nu - j_2^\nu$

$$\mathcal{L}' = -\frac{1}{4}F^2 + |D^{(+)}\phi_1|^2 + |D^{(-)}\phi_2|^2 - \begin{pmatrix}\phi_1^\star & \phi_2^\star\end{pmatrix}\begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$$

where $D_\nu^{(\pm)} = \partial_\nu \pm eA_\nu$

$\rightarrow$ no gauge invariance

2. Gauge field coupled to Noether current $j_1^\nu + j_2^\nu$

$$\mathcal{L}' = -\frac{1}{4}F^2 + |D^{(+)}\phi_1|^2 + |D^{(+)}\phi_2|^2 - \begin{pmatrix}\phi_1^\star & \phi_2^\star\end{pmatrix}\begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$$

$\rightarrow$ no consistent Maxwell equations $\partial_\rho\partial_\nu F^{\rho\nu} \neq 0$

# Constrained gauge invariance

Solution: *(i)* couple $A_\mu$ to Noether current
*(ii)* add covariant gauge fixing term $-(\partial_\nu A^\nu)^2/2\xi$
$\rightarrow$ Restriction to harmonic gauge functions $\Box f = 0$

Gauge field equation of motion

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi}\Box A^\nu = j_1^\nu + j_2^\nu$$

$\rightarrow$ gauge fixing condition

$$\frac{1}{\xi}\Box\partial_\nu A^\nu = \partial_\nu(j_1^\nu + j_2^\nu) = 2ie\mu^2(\phi_2^\star\phi_1 - \phi_1^\star\phi_2)$$

$\rightarrow$ gauge fixing at the classical level

# Conclusion

Main message:
Consistent interpretation of non-Hermitian Lagrangian

- Path integral quantisation *Alexandre, Ellis, Millington, Seynaeve - PRD 2018*
- Abelian Higgs mechanism *Alexandre, Ellis, Millington, Seynaeve - PRD 2019*
- Non-Abelian version (work in progress)
- Extension to non-Hermitian SUSY (work in progress)

# Appendix 1: Path integral

*Alexandre, Ellis, Millington, Seynaeve - PRD 2018*

- Define variables of integration $\Xi, \overline{\Xi}$ such that

$$\Phi^\dagger \begin{pmatrix} \Box + m_1^2 & \mu^2 \\ -\mu^2 & \Box + m_2^2 \end{pmatrix} \Phi = \overline{\Xi} \begin{pmatrix} \Box + M_+^2 & 0 \\ 0 & \Box + M_-^2 \end{pmatrix} \Xi$$

→ Gaussian integrals well defined

- Introduce sources $J, \overline{J}$ such that simultaneously

$$\frac{\delta S}{\delta \Xi} + \overline{J} = 0 \quad \text{and} \quad \frac{\delta S}{\delta \overline{\Xi}} + J = 0$$

→ no ambiguity in definition of saddle point

- Impose consistent $\mathcal{PT}$-symmetric properties for $J, \overline{J}$

→ partition function real: *(i)* depends on $\mu^4$ only
*(ii)* background gauge field real

# Appendix 2: P. Alternative approach

*Mannheim - PRD 2018*

Introduce a simlarity transformation

$$\phi_1 \to -i\phi_1 \qquad \text{and} \qquad \phi_1^\star \to i\phi_1^\star$$
$$\phi_2 \to -i\phi_2 \qquad \text{and} \qquad \phi_2^\star \to -i\phi_2^\star$$

$\to$ non-unitary transformation

Allows for Noether current to be conserved, but:
*(a)* kinetic term for $\phi_2$ changes sign
*(b)* consistent solution of vev equations requires analytical continuation of real and imaginary parts

# Appendix 3: Non-Hermitian fermion mass term

Original $\mathcal{PT}$-symmetric model *Bender, Jones, Rivers - PLB 2005*

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - m - \mu\gamma^5\right)\psi$$

- Dispersion relation $E^2 = \mathbf{p}^2 + m^2 - \mu^2$
- Conserved current and probabilty density *Alexandre, Bender - J. Phys. A 2015*

$$\begin{aligned} j^\nu &= \bar{\psi}\gamma^\nu\left(1 + \frac{\mu}{m}\gamma^5\right)\psi \\ \rho &= \left(1 + \frac{\mu}{m}\right)|\psi_R|^2 + \left(1 - \frac{\mu}{m}\right)|\psi_L|^2 \end{aligned}$$

$\rightarrow$ consistent with lattice simulations *Chernodub - J. Phys. A 2017*

- Gauged version *Alexandre, Bender, Millington - JHEP 2015*