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Webs in multiparton scattering

Work done with Gardi, Laenen, Magnea, Stavenga;
arXiv:0811.2067, arXiv:1008.0098, arXiv:1010.1860

Soft Gluons and New Physics at the LHC

Overview

What is the structure of soft gluon corrections in multiparton scattering?

- Overview of soft gluon resummation - webs.
- Path integral methods and the replica trick.
- Webs in multiparton scattering - mixing matrices.
- Applications and outlook.

Soft gluon radiation

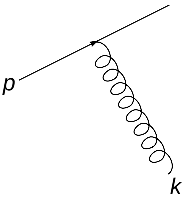
- If ξ is the energy carried by gluons, typically get cross-sections:

$$\frac{d\sigma}{d\xi} = \sum_{n,m} \alpha^n \left[c_{nm}^0 \frac{\log^m(\xi)}{\xi} + c_{nm}^1 \log^m(\xi) + \ldots \right]$$

- First set of terms corresponds to *eikonal approximation*, in which momenta $k_i \to 0$ for all (soft) emissions. Well understood.
- Second set of terms is *next-to-eikonal* (NE) limit i.e. first order in k_i.
- Perturbation theory breaks down $\Rightarrow$ need resummation.

Resummation of soft gluon logs

- Resummation of eikonal logs is well-known.
- Many different methods exists e.g. factorisation theorems (soft anomalous dimensions), SCET.
- I will summarise the *web* approach (Gatheral, Frenkel, Taylor, Sterman).
- First, note that in the eikonal approximation, the Feynman rules for gluon emission simplify.



$$\sim \frac{p^\mu}{p \cdot k}$$

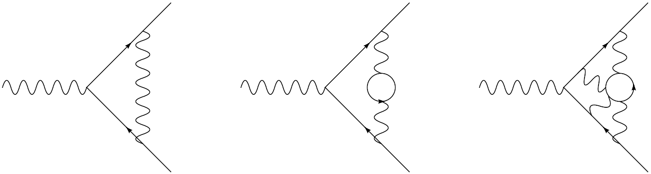
independently of the spin of the emitting particle!

Soft resummation - abelian case

- With these Feynman rules, only certain diagrams contribute.
- Can classify them at all orders in perturbation theory.
- In abelian theories, get a simple result (Yennie et. al.)

$$\mathcal{A} = \mathcal{A}_0 \exp\left[\sum G_c\right],$$

where $\mathcal{A}_0$ is the Born amplitude, and G_c are connected subdiagrams.



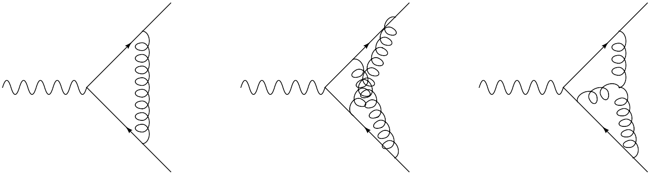
- Gives eikonal logarithms at all orders in α.

Soft resummation - nonabelian case

- Exponentiation generalisable to non-abelian theories, but structure is more complicated:

$$\mathcal{A} = \mathcal{A}_0 \exp\left[\sum \bar{C}_W W\right],$$

where W are *webs* (two-eikonal line irreducible subdiagrams).
- Webs have modified colour weights $\bar{C}_W$.



- More effort than abelian case, but still predicts eikonal logs to all orders.
- Only set up for two coloured particles...

Soft resummation - open problems

- Although some things are known about multileg processes, we would like to know more:

  What is the structure of webs for multiparton processes?

  (Gardi, Laenen, Stavenga, White; Mitov, Sterman, Sung)
- Another problem is what happens beyond the eikonal approximation (Grunberg et. al., Laenen et. al., Vogt et. al.).

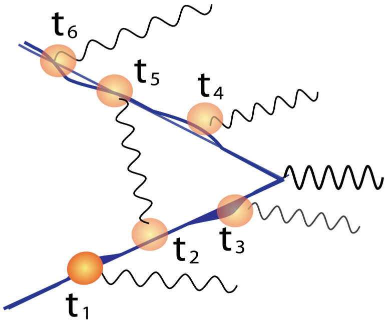
  Can we systematically classify next-to-eikonal logarithms?
- There has recently been progress on both these fronts.
- Here we focus on the first...

Motivation

- Q: “We already know how to resum some large logs in multiparton processes, so why do it using webs?”
- A: There are a number of motivations!
  1. Webs allow you to calculate the exponent of the soft gluon amplitude directly, thus makes extending resummation methods much easier.
  2. Webs contain *more information* than in other resummation approaches i.e. some finite terms also exponentiate, not just divergent logs.
  3. Webs allow us to classify soft gluon corrections beyond the eikonal approximation.
  4. Also more formal applications (e.g. sum over dipoles conjecture, $\mathcal{N} = 4$ SYM).
- Convenient to use path integral methods...

Path Integral Method for Soft Gluon Resummation

- Basic idea: rewrite a QCD scattering process in terms of *path integrals* over the emitting particle trajectories.

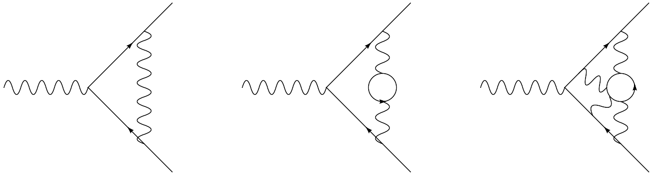


- Classical trajectories correspond to eikonal approximation (gluons have zero momentum).
- Expansion about classical trajectory gives NE corrections.
- Allows efficient classification of diagrams leading to large logs.

Soft photon field theory

- In the path integral approach, one obtains a field theory for the soft gauge field, which generates *subdiagrams* in the full theory.
- E.g. for an abelian theory, one finds the soft gluon generating functional

$$\mathcal{Z} = \int \mathcal{D}A_s^\mu e^{iS[A_s^\mu]} \prod_x \exp\left[i \int dx \cdot A_s(x) \right]$$
$$= \int \mathcal{D}A_s^\mu e^{iS[A_s^\mu]} \left[\Phi^{(1)} \otimes \Phi^{(2)} \otimes \cdots \otimes \Phi^{(L)} \right],$$



Soft gluon field theory

- Things are more complicated due to non-trivial colour structure.
- Soft gluon diagrams are generated by the generating functional

$$\mathcal{Z}_{JK} = \int \mathcal{D}A_s^{\mu} e^{iS[A_s^{\mu}]} \left\{ \prod_x \exp\left[i \int dx \cdot A_s(x) \right] \right\}_{JK}$$
$$= \int \mathcal{D}A_s^{\mu} e^{iS[A_s^{\mu}]} \left[\Phi^{(1)} \otimes \Phi^{(2)} \otimes \cdots \otimes \Phi^{(L)} \right]_{JK},$$

  where $\Phi^{(i)}$ is a Wilson line operator associated with parton i.
- J, K are indices in the space of possible colour flows.

Soft gluon field theory

- We can use the soft gluon field theory to classify all diagrams which contribute in the eikonal approximation (and beyond).
- In particular, we can show that diagrams *exponentiate*, as in the two line case.
- The diagrams which sit in the exponent are *multiparton webs*.
- They have interesting properties.
- Can prove exponentiation using the *replica trick* from statistical physics.

The replica trick

- The generating functional for soft gluon diagrams is

$$\mathcal{Z}_{JK} = \int \mathcal{D}A_s^\mu e^{iS[A_s^\mu]} \left[\Phi^{(1)} \otimes \Phi^{(2)} \otimes \cdots \otimes \Phi^{(L)}\right]_{JK},$$

- The replica trick tells us to consider a theory with N non-interacting copies of the soft gluons...

The replica trick

- The generating functional for the replicated theory is

$$\mathcal{Z}_{IJ}^N = \int \mathcal{D}A_\mu^1 \ldots \mathcal{D}A_\mu^N \, e^{i\sum_i S[A_\mu^i]} \left[(\Phi_1^{(1)}\Phi_2^{(1)} \ldots \Phi_N^{(1)}) \right. \\ \left. \otimes (\Phi_1^{(2)} \ldots \Phi_N^{(2)}) \otimes (\Phi_1^{(3)} \ldots \Phi_N^{(3)}) \ldots \otimes (\Phi_1^{(L)} \ldots \Phi_N^{(L)}) \right]_{IJ},$$

  where $\Phi_m^{(i)}$ is a Wilson line operator on parton line i, with replica number j.
- That is, each parton line has N Wilson line operators on it, ordered by increasing replica number.
- This generates all Feynman diagrams in the replicated theory, which may contain a single replica, or multiple replicas.

The replica trick

- By a simple mathematical identity

$$\mathcal{Z}^N = 1 + N \log \mathcal{Z} + \mathcal{O}(N^2).$$

- It follows immediately that

$$\mathcal{Z} = \exp\left[\sum_W W\right],$$

where W is any diagram which is linear in the number of replicas N.

- I.e. a subclass of soft gluon diagrams exponentiates in non-abelian theories, for any number of parton legs!
- In the replica language, these are the $\mathcal{O}(N)$ diagrams.
- These are our multiline webs.

- To find which diagrams are $\mathcal{O}(N)$, we need to read off the Feynman rules from the generating functional for the replicated theory.
- This is complicated by the fact that each line carries a product of Wilson lines

$$\Phi_1^{(i)} \dots \Phi_N^{(i)} = \mathcal{P} \exp \left[i g_s \int dt \, n_i \cdot A_1^{\mu} \right] \dots \mathcal{P} \exp \left[i g_s \int dt \, n_i \cdot A_N^{\mu} \right]$$

- To read off the Feynman rules, need to rewrite this as a single path-ordered exponential.

Replica ordering

- We can do this by introducing an operator $\mathcal{R}$, which orders gauge fields in terms of increasing replica number.
- Example:

$$\mathcal{R}[A_i^\mu A_j^\nu] = \begin{cases} A_i^\mu A_j^\nu, & i < j \\ A_j^\mu A_i^\nu, & i > j \end{cases}$$

- Then we can write the generating functional for the replicated theory as

$$\mathcal{Z}^N = \int \left[\mathcal{D}A_\mu^1\right] \ldots \left[\mathcal{D}A_\mu^N\right] \, e^{\mathrm{i}\sum_i S[A_\mu^i]} \, \mathcal{R} \Bigg\{$$

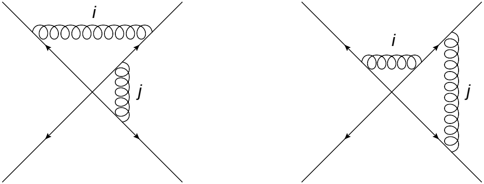
$$\mathcal{P}\exp\left[\mathrm{i}g_s \sum_{i=1}^{N} \int dt \, n_1 \cdot A^i\right] \otimes \ldots \otimes \mathcal{P}\exp\left[\mathrm{i}g_s \sum_{i=1}^{N} \int dt \, n_L \cdot A^i\right] \Bigg\}.$$

Feynman diagrams in the replicated theory

- All possible soft gluon subdiagrams are generated in the theory.
- Their colour factors are not the usual colour factors of QCD, as the replica ordering operator reorders the colour matrices.
- In the original theory, the eikonal amplitude thus consists of an exponential of those diagrams which are $\mathcal{O}(N)$ in the replicated theory, but where the colour factors are modified.
- For the two parton case, we find that these are exactly the webs of Gatheral, Frenkel and Taylor!
- For the multiparton case, things are more complicated...

Webs in multiparton scattering

- Consider the following two diagrams:



(a) (b)

- Contribution to the colour factors in the replicated theory are:

$$
\begin{array}{lcc}
 & \underline{\text{(a)}} & \underline{\text{(b)}} \\
i = j: & N\overline{C}(a) & N\overline{C}(b) \\
i < j: & \frac{1}{2}N(N-1)C(b) & \frac{1}{2}N(N-1)C(b) \\
i > j: & \frac{1}{2}N(N-1)C(a) & \frac{1}{2}N(N-1)C(a)
\end{array}
$$

Web example

- Taking the $\mathcal{O}(N)$ piece of these, we find that both diagrams contribute to the exponent of the eikonal scattering amplitude, with modified colour factors $\tilde{C}(a) = \frac{1}{2}[C(a) - C(b)]$, $\tilde{C}(b) = \frac{1}{2}[C(b) - C(a)]$.
- If $\mathcal{F}(a)$ is the kinematic part of diagram a etc., the total contribution to the exponent from these diagrams can be written

$$\left(\begin{array}{c} \mathcal{F}(a) \\ \mathcal{F}(b) \end{array} \right)^T \left(\begin{array}{c} \tilde{C}(a) \\ \tilde{C}(b) \end{array} \right) = \left(\begin{array}{c} \mathcal{F}(a) \\ \mathcal{F}(b) \end{array} \right)^T \frac{1}{2} \left(\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array} \right) \left(\begin{array}{c} C(a) \\ C(b) \end{array} \right).$$

- This structure is found to be quite general at higher loop order.

Webs in multiparton scattering

- In general, one finds closed sets of diagrams, related by permutations of gluons on the parton lines.
- The contribution of each set to the exponent of the eikonal scattering amplitude is

$$\sum_{D,D'} \mathcal{F}_D R_{DD'} C_{D'},$$

  where $R_{DD'}$ is a web-mixing matrix.
- The study of webs in multiparton scattering is equivalent to the study of these matrices!
- They have interesting properties...

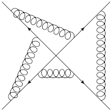
Web mixing matrices

- We observe the following interesting properties:
  1. Rows of web mixing matrices sum to zero i.e.

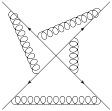
  $$\sum_{D'} R_{DD'} = 0.$$

     This is related to the fact that symmetric colour combinations do not exponentiate.
  2. The matrices are idempotent i.e. $R^2 = R$. This implies they have eigenvalues 0 and 1.
- These properties are intimately related to the cancellation of subdivergences in the exponent of the scattering amplitude.

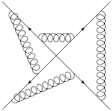
A four loop example



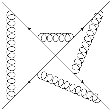
[[1,2],[3,1],[3,4],[2,4]]



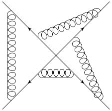
[[1,2],[2,3],[4,3],[4,1]]



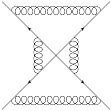
[[1,2],[3,2],[3,4],[4,1]]



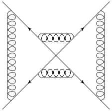
[[1,2],[2,3],[3,4],[1,4]]



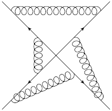
[[1,2],[3,2],[4,3],[1,4]]



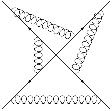
[[1,2],[1,3],[4,3],[4,2]]



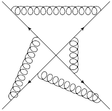
[[1,2],[3,2],[3,4],[1,4]]



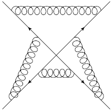
[[1,2],[1,3],[3,4],[4,2]]



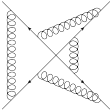
[[1,2],[3,1],[4,3],[4,2]]



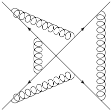
[[1,2],[1,3],[4,3],[2,4]]



[[1,2],[1,3],[3,4],[2,4]]



[[1,2],[2,3],[4,3],[1,4]]



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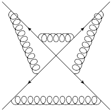
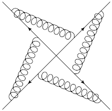
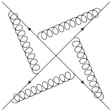


[[1,2],[3,2],[4,3],[4,1]]



[[1,2],[3,1],[4,3],[2,4]]



[[1,2],[2,3],[3,4],[4,1]]

Four loop mixing matrix ($\times 24$)

$$
\begin{pmatrix}
6 & -6 & 2 & 2 & -2 & 4 & -4 & 2 & -2 & -2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-6 & 6 & -2 & -2 & 2 & -4 & 4 & -2 & 2 & 2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 6 & -2 & 2 & 4 & -4 & -2 & 2 & -6 & 4 & 4 & -4 & -4 & 0 & 0 \\
2 & -2 & -2 & 6 & 2 & 4 & -4 & -2 & -6 & 2 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & 2 & 6 & 4 & -4 & -6 & -2 & -2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 2 & 2 & 2 & 4 & -4 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -2 & -2 & -2 & -4 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & -2 & -2 & -6 & -4 & 4 & 6 & 2 & 2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -6 & -2 & -4 & 4 & 2 & 6 & -2 & 4 & 4 & -4 & -4 & 0 & 0 \\
-2 & 2 & -6 & 2 & -2 & -4 & 4 & 2 & -2 & 6 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -2 & 2 & 0 & 0 & -2 & 2 & -2 & 4 & 0 & 0 & -4 & 0 & 0 \\
2 & -2 & 2 & -2 & -2 & 0 & 0 & 2 & 2 & -2 & 0 & 4 & -4 & 0 & 0 & 0 \\
-2 & 2 & -2 & 2 & 2 & 0 & 0 & -2 & -2 & 2 & 0 & -4 & 4 & 0 & 0 & 0 \\
2 & -2 & -2 & 2 & -2 & 0 & 0 & 2 & -2 & 2 & -4 & 0 & 0 & 4 & 0 & 0 \\
-18 & -6 & -6 & -6 & -18 & 12 & 12 & -6 & -18 & -18 & 12 & 12 & 12 & 12 & 24 & 0 \\
-6 & -18 & -18 & -18 & -6 & 12 & 12 & -18 & -6 & -6 & 12 & 12 & 12 & 12 & 0 & 24
\end{pmatrix}
$$

Summary

- Soft gluon corrections exponentiate for any number of parton legs.
- The exponent contains closed sets of diagrams, related by mixing matrices R.
- Each closed set is a "web", generalising the two-line results.
- The properties of the "web mixing matrix" R couple the colour and kinematic information of each diagram.

Other results

- Closed combinatoric formulae for the mixing matrices.
- Can also extend webs to next-to-eikonal order.
- For the two line case, agreement is found between the path integral methods and a traditional diagrammatic analysis (Laenen, Magnea, Stavenga, White).
- Results pave the way for resummation of multiline and / or NE effects.

The MSS approach

- Mitov, Sterman and Sung have also recently generalised webs to multiparton scattering.
- They obtain a combinatoric generalisation of Gatheral's formula, which is equivalent to that obtained from the path integral approach.
- They also consider renormalisation of webs, and find a non-trivial structure of nested counterterms at higher orders.
- This acts, with the mixing matrices R, to cancel all subdivergences in the exponent of the scattering amplitude.
- How this works in practice is under investigation...

Conclusions

- Path integral methods prove highly powerful in analysing soft gluon physics.
- New resummation results are obtained:
  1. Classification of multiline webs.
  2. Classification of NE logarithms.
- New mathematical structures have been found in the exponents of scattering amplitudes - mixing matrices R.
- Results pave the way for resumming multiline / NE effects in cross-sections (preliminary work in Drell-Yan).
- Also more formal applications?