Quantum Spectral Curve method in BFKL regime:
Asymptotic BFKL ansatz

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BFKL overview

- Summing leading logs in DIS at small $x$ or Regge limit of gluon scattering yields power-like behavior:

\[ \sigma \propto s^{\Delta - 1} \]

- LO same in QCD and N=4 SuperYang-Mills

- In SYM related to anomalous dim of single-trace operators at non-physical point of spin analytically continued to -1

- So one can use powerful methods of integrability in SYM to find BFKL eigenvalue

See integrability review [arXiv:1012.3982]
\( \mathcal{N}=4 \) SYM

- 4d supersymmetric conformal gauge theory
- The famous AdS/CFT correspondence:

\[
\mathcal{L} = -\frac{1}{2g_{YM}^2} \int d^4 x \, \text{tr} F^2 + \ldots
\]

- Potentially can be solved exactly in the planar limit: integrability
  \[
  \lambda = N_c g^2 = \text{const},
  \]
  \[
  N_c \to \infty, \ g \to 0
  \]
Integrability in SYM

- Infinite number of integrals of motion which can allow to solve a system with infinite number of d.o.f exactly
- Development: from discovery of perturbative integrability in the context of BFKL to Quantum Spectral Curve, calculating anomalous dimensions of finite-size operators at finite coupling

- Current status:
  - Spectral problem (2-pts) solved,
  - Huge progress in 3-pts
  (CFT → everything is defined by conf. dimensions and structure constants)

QSC: overview

- Ingredients: functions of complex spectral parameter $u$ with fixed analytic structure – quadratic cuts.

- Equations:
  - Algebraic constraints on Q-functions (“QQ-relations”)
  - Monodromy around the branch points

\[ g \equiv \sqrt{\lambda}/(4\pi) \to 0 \]

- Information about the state described, (i.e. the conformal dimension and other global charges) is encoded into the asymptotics at large $u$

BFKL and twist-2 operators in SYM

- Intercept is related to spin of twist-2 operators \( \text{Tr}(Z \nabla^S Z) \)

\[
  j(\Delta) = S + 2
\]

- Weak-coupling anomalous dimension has a singularity at \( S = -1 \)

To study the regime near the singularity, introduce a parameter

\[
  \Lambda \equiv \frac{g^2}{S+1}
\]

QSC simplifies near BFKL point

- Perturbation theory in parameter
  \[ \omega = S + 1 \to 0 \]
  which sums infinitely many orders of ordinary weak-coupling perturbation theory (leading singularity at each order in g)

Simplifications:
- Large asymptotics of certain functions at large values of spectral parameter \( u \) become non-negative: polynomial in the LO
- Some quantities become parametrically large/small in \( \omega \) as \( S \to -1 \)
Asymptotic BFKL Ansatz

- Using analytical properties of functions entering QSC, derive an ansatz for them - parametrization in terms of finite number of roots.

- QQ-relations → Bethe equations for roots, similar to Beisert-Eden-Staudacher Asymptotic Bethe Ansatz.

- Similarly to ABA, for twist $L$ operators it gives the anomalous dimension up to $L$-th order of perturbation theory. Useful for higher twists.

Tested for $L=3$ against [Beccaria, Macorini, Ratti, arXiv:1105.3577].

\[
\begin{align*}
\frac{Q_{1,1}^{+}}{Q_{1,1}^{-}} &= \frac{1}{m} \frac{B_{(+)}}{B_{(-)}} , \\
\frac{Q_{1,1}^{++}}{Q_{1,1}^{-}} &= -\frac{1}{(m\bar{m})^+} \frac{Q_{1,0}^{+}Q_{1,13}^+}{Q_{1,0}^{-}Q_{1,13}^-} , \\
\frac{Q_{1,1}^+}{Q_{1,1}^-} &= \frac{1}{m} \frac{R_{(+)}}{R_{(-)}} , \\
\frac{Q_{1,1}^-}{Q_{1,1}^+} &= \frac{2M}{(m\bar{m})^+} \left(\frac{x^+}{x^-}\right)^2 \left(\frac{\sigma^-}{\sigma^+}\right)^2 \frac{B_{1[0]1[13]}B_{4[0]}R_{4[24]}^+}{B_{1[0]1[13]}B_{4[0]}R_{4[24]}^-} = 1 , \\
\frac{(Q^{4,4})^+}{(Q^{4,4})^-} &= \frac{1}{m} \frac{R_{(+)}}{R_{(-)}} , \\
\frac{(Q^{4,4})^{++}}{(Q^{4,4})^{--}} &= -\frac{1}{(m\bar{m})^+} \frac{(Q^{4,4})^+}{(Q^{4,4})^-} \frac{(Q^{4,4})^{24}^+}{(Q^{4,4})^{24}^-} , \\
\frac{(Q^{4,4})^+}{(Q^{4,4})^-} &= \frac{1}{m} \frac{B_{(+)}}{B_{(-)}} .
\end{align*}
\]
LO BFKL

- Can be derived from our ABA: replace Bethe equations by an equivalent Baxter equation

\[
-2u^2 - \frac{1}{4} \left( 4c - \Delta^2 + \omega^2 - 1 \right) \right] Q(u) = \frac{1}{4} Q(u + i) \left[ (c + 2(u + i/2)^2)^2 - 4\Lambda \omega^3 + (u + i/2)^2 \omega^2 \right] + Q(u - i)
\]

- Baxter is solved in terms of hypergeometric functions, and the anomalous dimension $\gamma$ is found from the condition of “gluing” of a function on the two sides of the cut. Reproduce a well-know result

\[
-\frac{1}{\Lambda} = \psi \left( \frac{\gamma}{2} + 2 \right) + \psi \left( -\frac{\gamma}{2} \right) + 2\gamma_E
\]

\[
\gamma = -2\frac{g^2}{\omega} + 2\frac{g^4}{\omega^2} - 2\frac{g^8 (2\zeta(3) + 1)}{\omega^4} + 16g^{10} \frac{g^4}{\omega^5} + \ldots
\]

- Also reproduced in [arXiv:1408.2530] by directly solving QSC in LO
Double Logarithms

- Next singularity: summing all \((\alpha \ln^2 s)^k\) terms in scattering amplitudes
- Corresponds to twist-2 operators at \(\omega = S + 2 \to 0\)
- LO equation reproduced: \(\gamma (2\omega + \gamma) = -16g^2\)
- NLO and NNLO agree with weak coupling in [Marboe, Velizanin, Volin; arXiv:1412.4762v1]

\[
\gamma = g \left(-\frac{\omega}{g} + \sqrt{\left(\frac{\omega}{g}\right)^2 - 16}\right) + g^2 \frac{8\omega/g}{\sqrt{\left(\frac{\omega}{g}\right)^2 - 16}} + g^3 \frac{4(\pi^2 + 6) \left(\frac{\omega}{g}\right)^2 \left(\left(\frac{\omega}{g}\right)^2 - 20\right) + 256\pi^2}{3 \left(\left(\frac{\omega}{g}\right)^2 - 16\right)^{3/2}}
\]
Conclusions and further directions

- Integrability-based Quantum Spectral Curve method successfully applied to BFKL calculations
- LO BFKL eigenvalue reproduced
- Double-logarithmic equations: LO and the first three corrections
  - NLO BFKL from QSC is just a question of time. *What about NNLO?*